

# Local Theory for 2-Functors on Path 2-Groupoids

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## Abstract

In this article we discuss local aspects of 2-functors defined on the path 2-groupoid of a smooth manifold; in particular, local trivializations and descent data. This is a contribution to a project that provides an axiomatic formulation of connections on (possibly non-abelian) gerbes in terms of 2-functors. The main result of this paper establishes the first part of this formulation: we prove an equivalence between the globally defined 2-functors and their locally defined descent data. The second part appears in a separate publication; there we prove equivalences between descent data, on one side, and various existing versions of gerbes with connection on the other side.

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## 1 Introduction

The present article is a contribution to the axiomatic formulation of connections on (possibly non-abelian) gerbes carried out by the authors in several articles [SW09, SW11, SW]. It is based on 2-functors

$$F : \mathcal{P}_2(M) \longrightarrow T$$

defined on the path 2-groupoid of a smooth manifold  $M$ , with values in some “target” 2-category  $T$ . The path 2-groupoid  $\mathcal{P}_2(M)$  is a strict 2-groupoid with objects the points of  $M$ , 1-morphisms certain homotopy classes of paths, and 2-morphisms certain homotopy classes of homotopies between paths. A typical example of a target 2-category is the 2-category of algebras (over some fixed field), bimodules, and intertwiners. In that example, the algebra  $F(x) \in T$  associated to a point  $x \in M$  is supposed to be the *fibre* of the gerbe at  $x$ , the bimodule

$$F(\gamma) : F(x) \longrightarrow F(y)$$

associated to a path  $\gamma$  from  $x$  to  $y$  is supposed to be the *parallel transport* of the connection on that gerbe *along the path*  $\gamma$ , and the intertwiner

$$\begin{array}{ccc}
& F(\gamma_1) & \\
& \curvearrowright & \\
F(x) & \begin{array}{c} \parallel \\ F(\Sigma) \\ \parallel \end{array} & F(y) \\
& \curvearrowleft & \\
& F(\gamma_2) & 
\end{array}$$

associated to a homotopy  $\Sigma$  between paths  $\gamma_1, \gamma_2$  from  $x$  to  $y$  is supposed to be the *parallel transport* of the connection on that gerbe *along the surface*  $\Sigma$ .

In this article we discuss local properties of 2-functors defined on path 2-groupoids. Our main objective is to implement the notions of *structure 2-groupoids*, *local trivializations*, and *descent data* for a 2-functor  $F$ . Descent data plays a crucial role for specifying certain smoothness conditions for a 2-functor  $F$ .

In Section 2 we set up the basis for our discussion of local properties. In Section 2.1 we introduce the notion of a *local trivialization* of a 2-functor  $F : \mathcal{P}_2(M) \rightarrow T$  (Definition 2.1.1), which is central for this article. In order to say what a local trivialization is, we fix a strict 2-groupoid  $\text{Gr}$  and a 2-functor  $i : \text{Gr} \rightarrow T$ . Basically, a 2-functor  $F$  is locally  $i$ -trivializable if it factors locally through the 2-functor  $i$ . In more detail, we require that there exists a surjective submersion  $\pi : Y \rightarrow M$ , a strict 2-functor  $\text{triv} : \mathcal{P}_2(Y) \rightarrow \text{Gr}$ , and a natural equivalence

$$F \circ \pi_* \cong i \circ \text{triv}, \quad (1.1)$$

where  $\pi_* : \mathcal{P}_2(Y) \rightarrow \mathcal{P}_2(M)$  is the induced 2-functor on path 2-groupoids. The 2-functor  $i : \text{Gr} \rightarrow T$  plays the role of the typical fibre for  $F$ . Locally  $i$ -trivializable 2-functors form a 2-category  $\text{Funct}_i(\mathcal{P}_2(M), T)$ .

In Section 2.2 we introduce the notion of *descent data* for locally trivialized 2-functors. The descent data of a local trivialization consists of the 2-functor  $\text{triv} : \mathcal{P}_2(Y) \rightarrow \text{Gr}$  and further coherence data related to the natural equivalence (1.1). Descent data with respect to the structure 2-groupoid  $i : \text{Gr} \rightarrow T$  forms a 2-category  $\mathfrak{Des}^2(i)_M$ . In Section 2.3 we describe how to extract descent data from a given local trivialization of a 2-functor  $F$ .

Descent data plays a crucial role because it allows to impose *smoothness conditions*. We shall briefly outline these conditions – the full discussion is given in [SW]. We infer that the path 2-groupoid is a 2-groupoid internal to the category of *diffeological spaces* [SW11]. Diffeological spaces contain smooth manifolds as a full subcategory, but allow for many constructions which are obstructed for smooth manifolds. For example, the sets of 1-morphisms and 2-morphisms of the path 2-groupoid are quotients of subsets of mapping spaces between smooth manifolds, and all these operations lead to well-defined diffeological spaces. On the other side, however, the target 2-categories are typically not internal to the category of smooth manifolds or diffeological spaces, so that it is not possible to demand that  $F$  is smooth. Instead, we will assume that the structure 2-groupoid  $\text{Gr}$  is a *Lie 2-groupoid*, and impose smoothness conditions for the descent data of  $F$ , which is formulated

with respect to  $\text{Gr}$ . For instance, the first of these smoothness conditions is that the 2-functor  $\text{triv} : \mathcal{P}_2(Y) \longrightarrow \text{Gr}$  is smooth. There are further conditions related to the natural equivalence (1.1); these will be described in detail in [SW].

In Section 3 we establish a procedure that *reconstructs* a 2-functor from given descent data. This reconstruction procedure is, on a technical level, the main contribution of the present article. In order to explain some of the details, let  $\mathfrak{Des}_\pi^2(i) \subseteq \mathfrak{Des}^2(i)_M$  denote the 2-category of descent data with respect to a fixed surjective submersion  $\pi : Y \longrightarrow M$ . We introduce in Section 3.1 the *codescent 2-groupoid*  $\mathcal{P}_2^\pi(M)$  whose idea is to combine the path 2-groupoid of  $Y$  with additional “vertical jumps” in the fibres of  $\pi$ . The codescent 2-groupoid  $\mathcal{P}_2^\pi(M)$  serves two purposes. Firstly, we prove in Section 3.2 the existence of a 2-functor

$$s : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2^\pi(M)$$

that consistently lifts points, paths, and homotopies between paths from  $M$  to  $Y$ , by compensating the differences between local lifts with the vertical jumps. Secondly, we construct in Section 3.3 a “pairing 2-functor”

$$R : \mathfrak{Des}_\pi^2(i) \longrightarrow \text{Funct}(\mathcal{P}_2^\pi(M), T)$$

expressing the result that the codescent 2-groupoid  $\mathcal{P}_2^\pi(M)$  is “ $T$ -dual” to the descent 2-category. The composition of a 2-functor in the image of  $R$  with  $s$  results in a locally  $i$ -trivializable 2-functor on  $\mathcal{P}_2(M)$ , and this defines the reconstruction of a 2-functor from descent data.

In Section 4 we prove the main result of this article, namely that the correspondence between *globally defined* locally  $i$ -trivializable 2-functors  $F : \mathcal{P}_2(M) \longrightarrow T$  and *locally defined* descent data is one-to-one. More precisely, we prove (Theorem 4.2.2) that extraction and reconstruction establish an equivalence

$$\mathfrak{Des}^2(i)_M \cong \text{Funct}_i(\mathcal{P}_2(M), T) \tag{1.2}$$

between the 2-category  $\mathfrak{Des}^2(i)_M$  of descent data with respect to the structure 2-groupoid  $i : \text{Gr} \longrightarrow T$  and the 2-category  $\text{Funct}_i(\mathcal{P}_2(M), T)$  of locally  $i$ -trivializable 2-functors.

In order to explain the importance of this result let us return to the anticipated discussion of smoothness conditions that we impose on the descent data. Let us denote by  $\mathfrak{Des}^2(i)_M^\infty$  the sub-2-category of  $\mathfrak{Des}^2(i)_M$  that consists of *smooth descent data*. The equivalence (1.2) from the main result of the present article restricts to an equivalence between

$\mathfrak{Des}^2(i)_M^\infty$  and a sub-2-category of  $\text{Funct}_i(\mathcal{P}_2(M), T)$ , called the 2-category of *transport 2-functors on  $M$  with Gr-structure*. These transport 2-functors constitute our axiomatic formulation of connections on gerbes. In this interpretation, our main theorem implies an equivalence between transport 2-functors and local, smooth, differential-geometric data.

We have included an appendix containing a brief summary of the notions and conventions from higher category theory that we use.

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## 2 Locally Trivial 2-Functors and their Descent Data

In this section we introduce the central notions of the local theory of 2-functors.

### 2.1 Local Trivializations

Let  $M$  be a smooth manifold. For points  $x, y \in M$ , a *path*  $\gamma : x \longrightarrow y$  is a smooth map  $\gamma : [0, 1] \longrightarrow M$  with  $\gamma(0) = x$  and  $\gamma(1) = y$ . We require paths to have “sitting instants”, i.e. to be locally constant around  $\{0, 1\}$ . A *bigon*  $\Sigma : \gamma \Longrightarrow \gamma'$  between two paths  $\gamma, \gamma' : x \longrightarrow y$  is a smooth fixed-ends homotopy from  $\gamma$  to  $\gamma'$  with sitting instants at  $\gamma$  and  $\gamma'$ . The *path 2-groupoid*  $\mathcal{P}_2(M)$  [SW11] of a smooth manifold  $M$  is a strict 2-groupoid with

- (i) objects: the points of  $M$ .
- (ii) 1-morphisms: rank-one homotopy classes of paths.
- (iii) 2-morphisms: rank-two homotopy classes of bigons.

The process of taking classes by homotopies of certain rank is explained in [SW11]. For the purpose of this article, it suffices to accept that these assure the existence of strict, associative compositions and of strict inverses. For definitions and conventions related to 2-categories we refer to Appendix A. If  $f : M \longrightarrow N$  is a smooth map, we get a 2-functor

$f_* : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2(N)$ . For composable smooth maps  $f$  and  $g : N \longrightarrow O$  we get

$$(g \circ f)_* = g_* \circ f_*. \quad (2.1.1)$$

In this article, we study 2-functors

$$F : \mathcal{P}_2(M) \longrightarrow T \quad (2.1.2)$$

for  $T$  some 2-category called *target 2-category*. The 2-category of 2-functors (2.1.2) is denoted  $\text{Func}(\mathcal{P}_2(M), T)$ . If  $f : M \longrightarrow N$  is a smooth map, the composition of a 2-functor  $F : \mathcal{P}_2(N) \longrightarrow T$  with the 2-functor  $f_* : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2(N)$  is denoted by

$$f^*F := F \circ f_* : \mathcal{P}_2(M) \longrightarrow T.$$

Local trivializations of a 2-functor  $F : \mathcal{P}_2(M) \longrightarrow T$  have three attributes:

- (i) A strict 2-groupoid  $\text{Gr}$ , the *structure 2-groupoid*.
- (ii) A 2-functor  $i : \text{Gr} \longrightarrow T$  that indicates how the structure 2-groupoid is realized in the target 2-category.
- (iii) A surjective submersion  $\pi : Y \longrightarrow M$  implementing locality.

For a surjective submersion  $\pi : Y \longrightarrow M$  the fibre products  $Y^{[k]} := Y \times_M \dots \times_M Y$  are again smooth manifolds, and the canonical projections  $\pi_{i_1 \dots i_p} : Y^{[k]} \longrightarrow Y^{[p]}$  to the indexed factors are smooth maps.

**Definition 2.1.1.** A  $\pi$ -local  $i$ -trivialization of a 2-functor

$$F : \mathcal{P}_2(M) \longrightarrow T$$

is a pair  $(\text{triv}, t)$  of a strict 2-functor  $\text{triv} : \mathcal{P}_2(Y) \longrightarrow \text{Gr}$  and of a pseudonatural equivalence

$$\begin{array}{ccc} \mathcal{P}_2(Y) & \xrightarrow{\pi_*} & \mathcal{P}_2(M) \\ \text{triv} \downarrow & \swarrow t & \downarrow F \\ \text{Gr} & \xrightarrow{i} & T. \end{array}$$

In other words, a 2-functor  $F$  is locally trivializable, if its pullback  $\pi^*F$  to the covering space factorizes – up to pseudonatural equivalence – through the structure 2-groupoid  $\text{Gr}$ . A 2-functor is called *i-trivializable*, if it has a  $\text{id}_M$ -local *i*-trivialization. A 2-functor  $\text{triv} : \mathcal{P}_2(M) \longrightarrow \text{Gr}$  is called *i-trivial*, in which case we write  $\text{triv}_i := i \circ \text{triv}$  in order to abbreviate the notation. We also remark that by “pseudonatural equivalence” we mean a pseudonatural transformation *together* with a weak inverse and two invertible modifications expressing the invertibility (see Appendix A).

We define a 2-category  $\text{Triv}_\pi^2(i)$  of 2-functors with  $\pi$ -local *i*-trivialization: an object is a triple  $(F, \text{triv}, t)$  of a 2-functor  $F : \mathcal{P}_2(M) \longrightarrow T$  together with a  $\pi$ -local *i*-trivialization  $(\text{triv}, t)$ . A 1-morphism

$$(F, \text{triv}, t) \longrightarrow (F', \text{triv}', t')$$

is just a pseudonatural transformation  $F \longrightarrow F'$  between the two 2-functors (ignoring the trivialization), and a 2-morphism is just a modification between those.

## 2.2 Descent Data

Let  $i : \text{Gr} \longrightarrow T$  be a 2-functor from a strict 2-groupoid  $\text{Gr}$  to a 2-category  $T$ , and let  $\pi : Y \longrightarrow M$  be a surjective submersion. In the following three definitions, we define a 2-category  $\mathfrak{Des}_\pi^2(i)$  of *descent data*.

**Definition 2.2.1.** A descent object is a tuple  $(\text{triv}, g, \psi, f)$  consisting of

- (i) a strict 2-functor  $\text{triv} : \mathcal{P}_2(Y) \longrightarrow \text{Gr}$
- (ii) a pseudonatural equivalence  $g : \pi_1^* \text{triv}_i \longrightarrow \pi_2^* \text{triv}_i$
- (iii) an invertible modification  $\psi : \text{id}_{\text{triv}_i} \Longrightarrow \Delta^* g$
- (iv) an invertible modification  $f : \pi_{23}^* g \circ \pi_{12}^* g \Longrightarrow \pi_{13}^* g$

such that the diagrams

$$\begin{array}{ccc} \text{id}_{\pi_2^* \text{triv}_i} \circ g & \xrightarrow{\pi_2^* \psi \circ \text{id}} & \Delta_{22}^* g \circ g \\ & \searrow r & \swarrow \Delta_{122}^* f \\ & g & \end{array}, \quad \begin{array}{ccc} g \circ \text{id}_{\pi_1^* \text{triv}_i} & \xrightarrow{\text{id} \circ \pi_1^* \psi} & g \circ \Delta_{11}^* g \\ & \searrow l & \swarrow \Delta_{112}^* f \\ & g & \end{array} \quad (2.2.1)$$

and

$$\begin{array}{ccc}
& (\pi_{34}^*g \circ \pi_{23}^*g) \circ \pi_{12}^*g & \\
a \swarrow & & \searrow \pi_{234}^*f \circ \text{id} \\
\pi_{34}^*g \circ (\pi_{23}^*g \circ \pi_{12}^*g) & & \pi_{24}^*g \circ \pi_{12}^*g \\
\text{id} \circ \pi_{123}^*f \searrow & & \swarrow \pi_{124}^*f \\
\pi_{34}^*g \circ \pi_{13}^*g & \xrightarrow{\text{id} \circ \pi_{134}^*f} & \pi_{14}^*g.
\end{array} \tag{2.2.2}$$

are commutative. A descent object  $(\text{triv}, g, \psi, f)$  is called normalized, if the conditions

$$\text{id}_{\text{triv}_i} = \Delta^*g \quad \text{and} \quad g \circ \Delta_{21}^*g = \text{id}_{\pi_1^*\text{triv}_i}$$

and

$$\psi = \text{id}_{\Delta^*g} \quad \text{and} \quad \Delta_{121}^*f = \text{id}_{\Delta_{11}^*g}$$

hold.

In these diagrams,  $r$ ,  $l$  and  $a$  are the right and left unifiers and the associator of the 2-category  $T$ . Further is  $\Delta : Y \longrightarrow Y^{[2]}$  the diagonal map,  $\Delta_{112}, \Delta_{122} : Y^{[2]} \longrightarrow Y^{[3]}$  are the maps that duplicate the first and the second factor, respectively,  $\Delta_{jj} := \Delta \circ \pi_j$ , and  $\Delta_{21} : Y^{[2]} \longrightarrow Y^{[2]}$  exchanges the two components. Normalized descent objects play an important role in the discussion of surface holonomy; see Lemma 3.3.4 and [SW, Section 5].

**Definition 2.2.2.** Let  $(\text{triv}, g, \psi, f)$  and  $(\text{triv}', g', \psi', f')$  be descent objects. A descent 1-morphism  $(\text{triv}, g, \psi, f) \longrightarrow (\text{triv}', g', \psi', f')$  is a pair  $(h, \epsilon)$  of a pseudonatural transformation

$$h : \text{triv}_i \longrightarrow \text{triv}'_i$$

and an invertible modification

$$\epsilon : \pi_2^*h \circ g \Longrightarrow g' \circ \pi_1^*h$$



such that the diagrams

$$\begin{array}{ccc}
\pi_{23}^* g' \circ (\pi_2^* h \circ \pi_{12}^* g) & \xrightarrow{a} & (\pi_{23}^* g' \circ \pi_2^* h) \circ \pi_{12}^* g \\
\text{id} \circ \pi_{12}^* \epsilon \Downarrow & & \Downarrow \pi_{23}^* \epsilon^{-1} \text{id} \\
\pi_{23}^* g' \circ (\pi_{12}^* g' \circ \pi_1^* h) & & (\pi_3^* h \circ \pi_{23}^* g) \circ \pi_{12}^* g \\
a^{-1} \Downarrow & & \Downarrow a \\
(\pi_{23}^* g' \circ \pi_{12}^* g') \circ \pi_1^* h & & \pi_3^* h \circ (\pi_{23}^* g \circ \pi_{12}^* g) \\
f' \circ \text{id} \Downarrow & & \Downarrow \text{id} \circ f \\
\pi_{13}^* g' \circ \pi_1^* h & \xrightarrow{\pi_{13}^* \epsilon} & \pi_3^* h \circ \pi_{13}^* g.
\end{array} \tag{2.2.3}$$

and

$$\begin{array}{ccc}
\text{id}_{\text{triv}'_i} \circ h & \xrightarrow{l_h} & h \xrightarrow{r_h^{-1}} h \circ \text{id}_{\text{triv}_i} \\
\psi' \circ \text{id}_h \Downarrow & & \Downarrow \text{id}_h \circ \psi \\
\Delta^* g' \circ h & \xrightarrow{\Delta^* \epsilon} & h \circ \Delta^* g.
\end{array} \tag{2.2.4}$$

are commutative.

Finally, we introduce

**Definition 2.2.3.** Let  $(h_1, \epsilon_1)$  and  $(h_2, \epsilon_2)$  be descent 1-morphisms from a descent object  $(\text{triv}, g, \psi, f)$  to another descent object  $(\text{triv}', g', \psi', f')$ . A descent 2-morphism  $(h_1, \epsilon_1) \Rightarrow (h_2, \epsilon_2)$  is a modification

$$E : h_1 \Rightarrow h_2$$

such that the diagram

$$\begin{array}{ccc}
g' \circ \pi_1^* h_1 & \xrightarrow{\epsilon_1} & \pi_2^* h_1 \circ g \\
\text{id} \circ \pi_1^* E \Downarrow & & \Downarrow \pi_2^* E \circ \text{id} \\
g' \circ \pi_1^* h_2 & \xrightarrow{\epsilon_2} & \pi_2^* h_2 \circ g.
\end{array} \tag{2.2.5}$$

is commutative.

Descent objects, 1-morphisms and 2-morphisms form a 2-category  $\mathfrak{Des}_\pi^2(i)$  in an evident way. We remark that this 2-category comes with a strict 2-functor

$$V : \mathfrak{Des}_\pi^2(i) \longrightarrow \text{Func}(\mathcal{P}_2(Y), T).$$

From a descent object  $(\text{triv}, g, \psi, f)$  it keeps only the 2-functor  $\text{triv}$  and from a descent 1-morphism  $(h, \epsilon)$  only the pseudonatural transformation  $h$ .

**Example 2.2.4.** Let us briefly consider the 2-category  $\mathfrak{Des}_\pi^2(i)$  for the particular case that the manifolds  $M$  and  $Y$  are just points. Let  $\mathfrak{C}$  be a tensor category, and let  $\mathcal{B}\mathfrak{C}$  be the 2-category with one object associated to  $\mathfrak{C}$ , see Example A.2. Let  $\text{Gr}$  be the trivial 2-groupoid (one object, one 1-morphism and one 2-morphism), and let  $i : \text{Gr} \longrightarrow \mathcal{B}\mathfrak{C}$  be the 2-functor that sends the unique 1-morphism to the tensor unit in  $\mathfrak{C}$ . Then, a descent object is precisely a *special symmetric Frobenius algebra* object in  $\mathfrak{C}$ .

### 2.3 Extraction of Descent Data

We have so far introduced a 2-category  $\text{Triv}_\pi^2(i)$  of 2-functors with  $\pi$ -local  $i$ -trivializations and a 2-category  $\mathfrak{Des}_\pi^2(i)$  of descent data, both associated to a surjective submersion  $\pi$  and a 2-functor  $i : \text{Gr} \longrightarrow T$ . Now we define a 2-functor

$$\text{Ex}_\pi : \text{Triv}_\pi^2(i) \longrightarrow \mathfrak{Des}_\pi^2(i)$$

between these 2-categories. This 2-functor *extracts* descent data from 2-functors with local trivializations.

Let  $F : \mathcal{P}_2(M) \longrightarrow T$  be a 2-functor with a  $\pi$ -local  $i$ -trivialization  $(\text{triv}, t)$ . We recall that by our conventions the pseudonatural equivalence  $t$  comes with a weak inverse  $\bar{t} : \text{triv}_i \longrightarrow \pi^*F$  and with invertible modifications

$$i_t : \bar{t} \circ t \Longrightarrow \text{id}_{\pi^*F} \quad \text{and} \quad j_t : \text{id}_{\text{triv}_i} \Longrightarrow t \circ \bar{t} \quad (2.3.1)$$

satisfying the identities (A.1). We define a pseudonatural equivalence

$$g : \pi_1^* \text{triv}_i \longrightarrow \pi_2^* \text{triv}_i$$

as the composition  $g := \pi_2^* t \circ \pi_1^* \bar{t}$  of pseudonatural equivalences. This composition is well-defined since  $\pi_1^* \pi^* F = \pi_2^* \pi^* F$ . We obtain  $\Delta^* g = t \circ \bar{t}$ , so that the definition  $\psi := j_t$  yields an invertible modification

$$\psi : \text{id}_{\text{triv}_i} \Longrightarrow \Delta^* g.$$

Finally, we define an invertible modification

$$f : \pi_{23}^* g \circ \pi_{12}^* g \Longrightarrow \pi_{13}^* g$$

as the composition

$$\begin{aligned}
(\pi_3^* t \circ \pi_2^* \bar{t}) \circ (\pi_2^* t \circ \pi_1^* \bar{t}) &\Longrightarrow \pi_3^* t \circ ((\pi_2^* \bar{t} \circ \pi_2^* t) \circ \pi_1^* \bar{t}) \\
&\Downarrow \text{id} \circ (\pi_2^* i_t \circ \text{id}) \\
\pi_3^* t \circ (\text{id}_{\pi^* F} \circ \pi_1^* \bar{t}) &\xrightarrow{\text{id} \circ r_{\pi_1^* \bar{t}}} \pi_3^* t \circ \pi_1^* \bar{t}
\end{aligned}$$

where  $r$  is the right unifier of  $\text{Func}(\mathcal{P}_2(Y^{[2]}), T)$ , and the first arrow summarizes two obvious occurrences of associators.

**Lemma 2.3.1.** *The modifications  $\psi$  and  $f$  make the diagrams (2.2.1) and (2.2.2) commutative, so that*

$$\text{Ex}_\pi(F, \text{triv}, t) := (\text{triv}, g, \psi, f)$$

*is a descent object.*

*Proof.* We prove the commutativity of the diagram on the left hand side of (2.2.1) by patching it together from commutative diagrams:

The six subdiagrams are commutative: A is the Pentagon axiom (C4) of  $T$ , B's are the naturality of the associator, C and D are diagrams that follow from the coherence theorem for the 2-category  $T$ , and the remaining small triangle is axiom (C2). The commutativity of the second diagram in (2.2.1) and the one of diagram (2.2.2) can be shown in the same way.  $\square$

Now let  $A : F \longrightarrow F'$  be a pseudonatural transformation between two 2-functors with  $\pi$ -local  $i$ -trivializations  $t : \pi^* F \longrightarrow \text{triv}_i$  and  $t' : \pi^* F' \longrightarrow \text{triv}'_i$ . Let  $i_t, j_t$  and  $i_{t'}, j_{t'}$  be the modifications (2.3.1) we have chosen for the weak inverses  $\bar{t}$  and  $\bar{t}'$ . We define a

pseudonatural transformation

$$h : \text{triv}_i \longrightarrow \text{triv}'_i$$

by  $h := (t' \circ \pi^* A) \circ \bar{t}$ , and an invertible modification  $\epsilon$  by

$$\begin{aligned} \pi_2^* h \circ g &\Longrightarrow (\pi_2^* t' \circ \pi_2^* \pi^* A) \circ ((\pi_2^* \bar{t} \circ \pi_2^* t) \circ \pi_1^* \bar{t}) \\ &\Downarrow (\pi_2^* l_{t'}^{-1} \circ \text{id}) \circ (\pi_2^* i_t \circ \text{id}) \\ &((\pi_2^* t' \circ \text{id}) \circ \pi_2^* \pi^* A) \circ (\text{id} \circ \pi_1^* \bar{t}) \\ &\Downarrow ((\text{id} \circ \pi_1^* i_{t'}^{-1}) \circ \text{id}) \circ \pi_1^* r_t \\ &((\pi_2^* t' \circ (\pi_1^* \bar{t} \circ \pi_1^* t')) \circ \pi_1^* \pi^* A) \circ \pi_1^* \bar{t} \Longrightarrow g' \circ \pi_1^* h. \end{aligned}$$

Here, the unlabelled arrows summarize the definitions of  $h$  and  $g$  and several obvious occurrences of associators. Arguments similar to those given in the proof of Lemma 2.3.1 show the following lemma.

**Lemma 2.3.2.** *The modification  $\epsilon$  makes the diagrams (2.2.3) and (2.2.4) commutative, so that  $\text{Ex}_\pi(A) := (h, \epsilon)$  is a descent 1-morphism*

$$\text{Ex}_\pi(A) : \text{Ex}_\pi(F) \longrightarrow \text{Ex}_\pi(F').$$

In order to continue the definition of the 2-functor  $\text{Ex}_\pi$  we consider a modification  $B : A_1 \Longrightarrow A_2$  between pseudonatural transformations  $A_1, A_2 : F \longrightarrow F'$  of 2-functors with  $\pi$ -local  $i$ -trivializations  $t : \pi^* F \longrightarrow \text{triv}_i$  and  $t' : \pi^* F' \longrightarrow \text{triv}'_i$ . Let  $(h_k, \epsilon_k) := \text{Ex}_\pi(A_k)$  be the associated descent 1-morphisms for  $k = 1, 2$ . We define a modification  $E : h_1 \Longrightarrow h_2$  by

$$h_1 = (t' \circ \pi^* A_1) \circ \bar{t} \xrightarrow{(\text{id} \circ \pi^* B) \circ \text{id}} (t' \circ \pi^* A_2) \circ \bar{t} = h_2.$$

**Lemma 2.3.3.** *The modification  $E$  makes the diagram (2.2.5) commutative so that  $\text{Ex}_\pi(B) := E$  is a descent 2-morphism*

$$\text{Ex}_\pi(B) : \text{Ex}_\pi(A_1) \Longrightarrow \text{Ex}_\pi(A_2).$$

In order to finish the definition of the 2-functor  $\text{Ex}_\pi$  we have to define its compositors and unitors. We consider two composable pseudonatural transformations  $A_1 : F \longrightarrow F'$  and  $A_2 : F' \longrightarrow F''$  and the extracted descent 1-morphisms  $(h_k, \epsilon_k) := \text{Ex}_\pi(A_k)$  for  $k = 1, 2$  and  $(\tilde{h}, \tilde{\epsilon}) := \text{Ex}_\pi(A_2 \circ A_1)$ . The compositor

$$c_{A_1, A_2} : \text{Ex}_\pi(A_2) \circ \text{Ex}_\pi(A_1) \Longrightarrow \text{Ex}_\pi(A_2 \circ A_1)$$

is the modification  $h_2 \circ h_1 \Rightarrow \tilde{h}$  defined by

$$\begin{aligned} ((t'' \circ \pi^* A_2) \circ \bar{t}') \circ ((t' \circ \pi^* A_1) \circ \bar{t}) &\Rightarrow (t'' \circ (\pi^* A_2 \circ ((\bar{t}' \circ t') \circ \pi^* A_1))) \circ \bar{t} \\ &\Downarrow (\text{id} \circ (\text{id} \circ (i_{t'} \circ \text{id}))) \circ \text{id} \\ (t'' \circ (\pi^* A_2 \circ (\text{id} \circ \pi^* A_1))) \circ \bar{t} &\Longrightarrow (t'' \circ \pi^* (A_2 \circ A_1)) \circ \bar{t}. \end{aligned}$$

For a 2-functor  $F : \mathcal{P}_2(M) \longrightarrow T$  we find  $\text{Ex}_\pi(\text{id}_F) = t \circ \bar{t}$ . So, the unitor

$$u_F : \text{Ex}_\pi(\text{id}_F) \Rightarrow \text{id}_{\text{triv}_i}$$

is the modification  $u_F := j_t^{-1}$ . The identities (A.1) for  $i_t$  and  $j_t$  show that compositors and unitors are descent 2-morphisms. The following statement is now straightforward to check.

**Proposition 2.3.4.** *The structure collected above furnishes a 2-functor*

$$\text{Ex}_\pi : \text{Triv}_\pi^2(i) \longrightarrow \mathfrak{Des}_\pi^2(i).$$

### 3 Reconstruction from Descent Data

We have so far described how globally defined 2-functors induce locally defined structure, in terms of the 2-functor  $\text{Ex}_\pi$ . In this section we describe a 2-functor going in the other direction.

#### 3.1 A Covering of the Path 2-Groupoid

In this section we introduce the *codescent 2-groupoid*  $\mathcal{P}_2^\pi(M)$  associated to a surjective submersion  $\pi : Y \longrightarrow M$ . It combines the path 2-groupoid of  $Y$  with additional jumps between the fibres. This construction generalizes the groupoid  $\mathcal{P}_1^\pi(M)$  from [SW09].

The objects of  $\mathcal{P}_2^\pi(M)$  are all points  $a \in Y$ . There are two “basic” 1-morphisms:

- (1) *Paths*: rank-one homotopy classes of paths  $\gamma : a \longrightarrow a'$  in  $Y$ .
- (2) *Jumps*: points  $\alpha \in Y^{[2]}$  considered as 1-morphisms from  $\pi_1(\alpha)$  to  $\pi_2(\alpha)$ .

The set of 1-morphisms of  $\mathcal{P}_2^\pi(M)$  is freely generated from these two basic 1-morphisms, i.e. we have a formal composition  $*$  and a formal identity  $\text{id}_a^*$  (the empty composition) associated to every object  $a \in Y$ . We introduce six “basic” 2-morphisms:

(1) Four 2-morphisms of *essential* type:

- (a) Rank-two homotopy classes of bigons  $\Sigma : \gamma_1 \Longrightarrow \gamma_2$  in  $Y$  going between paths.
- (b) Rank-one homotopy classes of paths  $\Theta : \alpha \longrightarrow \alpha'$  in  $Y^{[2]}$  considered as 2-isomorphisms

$$\Theta : \alpha' * \pi_1(\Theta) \Longrightarrow \pi_2(\Theta) * \alpha,$$

going between 1-morphisms mixed from jumps and paths.

- (c) Points  $\Xi \in Y^{[3]}$  considered as 2-isomorphisms

$$\Xi : \pi_{23}(\Xi) * \pi_{12}(\Xi) \Longrightarrow \pi_{13}(\Xi)$$

going between jumps.

- (d) Points  $a \in Y$  considered as 2-isomorphisms

$$\Delta_a : \text{id}_a^* \Longrightarrow (a, a)$$

relating the formal identity with the trivial jump.

In (b) to (d) we demand that the 2-morphisms  $\Theta$ ,  $\Xi$  and  $\Delta_a$  come with formal inverses, denoted by  $\Theta^{-1}$ ,  $\Xi^{-1}$  and  $\Delta_a^{-1}$ .

(2) Two 2-morphisms of *technical* type:

- (a) associators for the formal composition, i.e. 2-isomorphisms

$$a_{\beta_1, \beta_2, \beta_3}^* : (\beta_3 * \beta_2) * \beta_1 \Longrightarrow \beta_3 * (\beta_2 * \beta_1)$$

for  $\beta_k$  either paths or jumps, and unifiers

$$l_\beta : \beta * \text{id}_a^* \Longrightarrow \beta \quad \text{and} \quad r_\beta : \text{id}_b^* * \beta \Longrightarrow \beta.$$

- (b) for points  $a \in Y$  and composable paths  $\gamma_1$  and  $\gamma_2$  2-isomorphisms

$$u_a^* : \text{id}_a \Longrightarrow \text{id}_a^* \quad \text{and} \quad c_{\gamma_1, \gamma_2}^* : \gamma_2 * \gamma_1 \Longrightarrow \gamma_2 \circ \gamma_1$$

expressing that the formal composition restricted to paths yields the usual composition of paths.

Now we consider the set which is freely generated from these basic 2-morphisms in virtue of a formal horizontal composition  $*$  and a formal vertical composition  $\otimes$ . The formal identity 2-morphisms are denoted by  $\text{id}_\beta^\otimes : \beta \Longrightarrow \beta$  for any 1-morphism  $\beta$ . The set of 2-morphisms of the 2-category  $\mathcal{P}_2^\pi(M)$  is this set subject to the following list of identifications:

- (I) Identifications of *2-categorical type*. The formal compositions  $*$  and  $\otimes$ , and the 2-isomorphisms of type (2a) form the structure of a 2-category and we impose all identifications required by the axioms (C1) to (C4).
- (II) Identifications of *2-functorial type*. We have the structure of a 2-functor

$$\iota : \mathcal{P}_2(Y) \longrightarrow \mathcal{P}_2^\pi(M).$$

This 2-functor regards points, paths and bigons in  $Y$  as objects, 1-morphisms of type (1) and 2-morphisms of type (1a), respectively. Its compositors and unitors are the 2-isomorphisms  $c^*$  and  $u^*$  of type (2b). We impose all identification required by the axioms (F1) to (F4) for this 2-functor.

- (III) Identifications of *transformation type*. We have the structure of a pseudonatural transformation

$$\Gamma : \pi_1^* \iota \longrightarrow \pi_2^* \iota$$

between 2-functors defined over  $Y^{[2]}$ . Its component at a 1-morphism  $\Theta : \alpha \longrightarrow \alpha'$  in  $\mathcal{P}_1(Y^{[2]})$  is the 2-isomorphism  $\Theta$  of type (1b). We impose all identifications required by the axioms (T1) and (T2) for this pseudonatural transformation.

- (IV) Identification of *modification type*. We have the structure of a modification

$$\pi_{23}^* \Gamma \circ \pi_{12}^* \Gamma \Longrightarrow \pi_{13}^* \Gamma \tag{3.1.1}$$

between pseudonatural transformations of 2-functors defined over  $Y^{[3]}$ . Its component at an object  $\Xi \in Y^{[3]}$  is the 2-isomorphism  $\Xi$  of type (1c). We have the structure of another modification

$$\text{id}_\iota \Longrightarrow \Delta^* \Gamma \tag{3.1.2}$$

between pseudonatural transformations of 2-functors over  $Y$ , whose component at an object  $a \in Y$  is the 2-isomorphism  $\Delta_a$  of type (1d). We impose all identifications required by the commutativity of diagram (A.4) for both modifications.

(V) Identifications of *essential type*:

1. For every point  $\Psi \in Y^{[4]}$  we impose the commutativity of the diagram

$$\begin{array}{ccccc}
 & & (\pi_{34}(\Psi) * \pi_{23}(\Psi)) * \pi_{12}(\Psi) & & \\
 & \swarrow a^* & & \searrow \pi_{234}(\Psi) * \text{id}^* & \\
 \pi_{34}(\Psi) * (\pi_{23}(\Psi) * \pi_{12}(\Psi)) & & & & \pi_{24}(\Psi) * \pi_{12}(\Psi) \\
 & \searrow \text{id}^* * \pi_{123}(\Psi) & & \swarrow \pi_{124}(\Psi) & \\
 & \pi_{34}(\Psi) * \pi_{13}(\Psi) & \xRightarrow{\pi_{134}(\Psi)} & \pi_{14}(\Psi) & 
 \end{array}$$

of compositions of jumps.

2. For every point  $\alpha \in Y^{[2]}$  we impose the commutativity of the diagrams

$$\begin{array}{ccc}
 \text{id}_b^* * \alpha & \xRightarrow{b * \text{id}_\alpha^*} & (b, b) * \alpha \\
 \searrow r_\alpha^* & & \swarrow (a, b, b) \\
 & \alpha & 
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 \alpha * \text{id}_a^* & \xRightarrow{\text{id}_\alpha^* \circ a} & \alpha * (a, a) \\
 \searrow l_\alpha^* & & \swarrow (a, a, b) \\
 & \alpha & 
 \end{array}$$

According to (I) we have defined a 2-category  $\mathcal{P}_2^\pi(M)$ .

**Lemma 3.1.1.** *The 2-category  $\mathcal{P}_2^\pi(M)$  is a 2-groupoid.*

*Proof.* All 2-morphisms except those of type (1a) are invertible by definition. But for a 2-morphism of type (1a), a bigon  $\Sigma : \gamma \Rightarrow \gamma'$ , we have

$$\Sigma^{-1} \circledast \Sigma \stackrel{\text{(II)}}{=} \Sigma^{-1} \bullet \Sigma = \text{id}_\gamma \stackrel{\text{(II)}}{=} \text{id}_\gamma^\circledast,$$

and analogously  $\Sigma \circledast \Sigma^{-1} = \text{id}_{\gamma'}^\circledast$ . Here we have used identification (II); more precisely axiom (F1) of the 2-functor  $\iota : \mathcal{P}_2(Y) \rightarrow \mathcal{P}_2^\pi(M)$ . To see that a path  $\gamma : a \rightarrow b$  is invertible, we claim that  $\gamma^{-1}$  is a weak inverse. It is easy to construct the 2-isomorphisms  $i_\gamma$  and  $j_\gamma$  using the 2-isomorphisms of type (2b). The required identities (A.1) for these 2-isomorphisms are then satisfied due to identification (II). To see that a jump  $\alpha \in Y^{[2]}$  with  $\alpha = (x, y)$  is invertible, we claim that  $\bar{\alpha} := (y, x)$  is a weak inverse. The 2-isomorphisms  $i_\alpha$  and  $j_\alpha$  can be constructed from 2-isomorphisms of types (1c) and (1d). The identities (A.1) are satisfied due to identifications (V1) and (V2).  $\square$



We remark that we have a 2-functor  $\iota : \mathcal{P}_2(Y) \hookrightarrow \mathcal{P}_2^\pi(M)$ , a pseudonatural transformation  $\Gamma$  and modifications (3.1.1) and (3.1.2) claimed by identifications (II), (III) and (IV).

## 3.2 The Section 2-Functor

There is a canonical strict 2-functor

$$p^\pi : \mathcal{P}_2^\pi(M) \longrightarrow \mathcal{P}_2(M)$$

whose composition with the 2-functor  $\iota$  is equal to the 2-functor  $\pi_* : \mathcal{P}_2(Y) \longrightarrow \mathcal{P}_2(M)$  induced from the projection, i.e.

$$p^\pi \circ \iota = \pi_*. \quad (3.2.1)$$

It sends all 1-morphisms and 2-morphisms which are not in the image of  $\iota$  to identities. In this section we show the following result.

**Proposition 3.2.1.** *The 2-functor  $p^\pi$  is an equivalence of 2-categories.*

In order to prove this we introduce an inverse 2-functor

$$s : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2^\pi(M).$$

Since the 2-functor  $p^\pi$  is surjective on objects, we call  $s$  the *section 2-functor*. To define  $s$ , we lift points, paths and bigons in  $M$  along the surjective submersion  $\pi$ , and use the jumps and the several 2-morphisms of the codescent 2-groupoid whenever no “global” lifts exist.

For preparation we need the following technical lemma.

**Lemma 3.2.2.** *Let  $\gamma : x \longrightarrow y$  be a path in  $M$ , and let  $\tilde{x}, \tilde{y} \in Y$  be lifts of the endpoints, i.e.  $\pi(\tilde{x}) = x$  and  $\pi(\tilde{y}) = y$ .*

- (a) *There exists a 1-morphism  $\tilde{\gamma} : \tilde{x} \longrightarrow \tilde{y}$  in  $\mathcal{P}_2^\pi(M)$  such that  $p^\pi(\tilde{\gamma}) = \gamma$ .*
- (b) *Let  $\tilde{\gamma} : \tilde{x} \longrightarrow \tilde{y}$  and  $\tilde{\gamma}' : \tilde{x} \longrightarrow \tilde{y}$  be two such 1-morphisms. Then, there exists a unique 2-isomorphism  $A : \tilde{\gamma} \Longrightarrow \tilde{\gamma}'$  in  $\mathcal{P}_2^\pi(M)$  such that  $p^\pi(A) = \text{id}_\gamma$ .*

The assertion (a) is proven as [SW09, Lemma 2.15]. The proof of (b) requires some preparation.

**Lemma 3.2.3.** *Let  $p \in M$  be a point and  $a, b \in Y$  with  $\pi(a) = \pi(b) = p$ . Let  $\alpha : a \longrightarrow b$  and  $\beta : a \longrightarrow b$  be 1-morphisms in  $\mathcal{P}_2^\pi(M)$  which are compositions of jumps.*

- (a) *There exists a 2-isomorphism  $\Xi : \alpha \Longrightarrow \beta$  with  $p^\pi(\Xi) = \text{id}_{\text{id}_p}$ .*
- (b) *Any 2-isomorphism  $\Xi : \alpha \Longrightarrow \beta$  with  $p^\pi(\Xi) = \text{id}_{\text{id}_p}$  can be represented by a composition of 2-morphisms of type (1c).*
- (c) *The 2-isomorphism from (a) is unique.*

Proof. It is easy to construct the 2-isomorphism of (a) using only 2-isomorphisms of type (1c) and their inverses. To show (b) let  $\Xi : \alpha \Longrightarrow \beta$  be a 2-isomorphism with  $p^\pi(\Xi) = \text{id}_{\text{id}_p}$ , represented by a composition of 2-morphisms of any type. In the following we draw pasting diagrams to demonstrate that all 2-morphisms of types (1a), (1b) and (1d) can subsequently be killed.

To prepare some machinery notice that identification (III) imposes axiom (T2) for the pseudonatural transformation  $\Gamma$ , which is, for any bigon  $\Sigma : \Theta_1 \Longrightarrow \Theta_2$  in  $Y^{[2]}$ , the identity:

$$\begin{array}{ccc}
 \begin{array}{c}
 \pi_1(\alpha) \xrightarrow{\pi_1(\Theta_1)} \pi_1(\alpha') \\
 \alpha \downarrow \quad \swarrow \Theta_1 \quad \downarrow \alpha' \\
 \pi_2(\alpha) \xrightarrow{\pi_2(\Theta_1)} \pi_2(\alpha') \\
 \quad \searrow \pi_2(\Sigma) \quad \swarrow \\
 \quad \pi_2(\Theta_2)
 \end{array}
 & = &
 \begin{array}{c}
 \pi_1(\Theta_1) \\
 \searrow \pi_1(\Sigma) \quad \swarrow \\
 \pi_1(\alpha) \xrightarrow{\pi_1(\Theta_2)} \pi_1(\alpha') \\
 \alpha \downarrow \quad \swarrow \Theta_2 \quad \downarrow \alpha' \\
 \pi_2(\alpha) \xrightarrow{\pi_2(\Theta_2)} \pi_2(\alpha')
 \end{array}
 \end{array} \tag{3.2.2}$$

In the same way, identification (IV) imposes the axiom for the modification  $\text{id}_\ell \Longrightarrow \Delta^* \Gamma$ , which is, for any path  $\gamma : a \longrightarrow b$  in  $Y$ , the identity

$$\begin{array}{ccc}
 \begin{array}{c}
 a \xrightarrow{\gamma} b \\
 \Delta(a) \swarrow \quad \searrow \Delta(b) \\
 \begin{array}{ccc}
 a & \xrightarrow{\gamma} & b \\
 \downarrow \text{id}_a^* & \swarrow \gamma^* & \downarrow \text{id}_b^* \\
 a & \xrightarrow{\gamma} & b
 \end{array}
 \end{array}
 & = &
 \begin{array}{c}
 a \xrightarrow{\gamma} b \\
 \Delta(a) \swarrow \quad \searrow \Delta(b) \\
 \begin{array}{ccc}
 a & \xrightarrow{\gamma} & b \\
 \downarrow \text{id}_a^* & \swarrow \gamma^* & \downarrow \text{id}_b^* \\
 a & \xrightarrow{\gamma} & b
 \end{array}
 \end{array}
 \end{array} \tag{3.2.3}$$

Using (3.2.3) we can write the identity 2-morphism associated to the path  $\gamma$  in a very fancy way, namely

$$\text{id}_\gamma^\otimes = a \begin{array}{c} \begin{array}{ccc} & \gamma & \\ \nearrow & \Downarrow & \searrow \\ & b & \\ \nwarrow & \Downarrow & \nearrow \\ & a & \\ \searrow & \Downarrow & \swarrow \\ & b & \end{array} \\ \end{array} b. \quad (3.2.4)$$

Now suppose that  $\Sigma : \gamma \Longrightarrow \gamma'$  is some 2-morphism of type (1a) that we want to kill. We write  $\Sigma = \Sigma \otimes \text{id}_\gamma^\otimes$  and use (3.2.4). Using the naturality of the 2-morphism  $l_\gamma^*$  claimed by identification (I) we have

$$\Sigma = a \begin{array}{c} \begin{array}{ccc} & \gamma & \\ \nearrow & \Downarrow & \searrow \\ & b & \\ \nwarrow & \Downarrow & \nearrow \\ & a & \\ \searrow & \Downarrow & \swarrow \\ & b & \end{array} \\ \end{array} b = a \begin{array}{c} \begin{array}{ccc} & \gamma & \\ \nearrow & \Downarrow & \searrow \\ & b & \\ \nwarrow & \Downarrow & \nearrow \\ & a & \\ \searrow & \Downarrow & \swarrow \\ & b & \end{array} \\ \end{array} b$$

where the second identity is obtained from (3.2.2) by taking  $\Theta_1 := \Delta(\gamma)$  and  $\Theta_2 := (\gamma, \gamma')$  which is only possible because we have assumed that  $p^\pi(\Sigma) = \text{id}_{\text{id}_p}$ . We can thus kill every 2-morphism of type (1a).

Suppose now that  $\Psi : \mu \Longrightarrow \nu$  is a 2-morphism of type (1b). To kill it we need identification (IV), namely the axiom for the modification  $\pi_{23}^* \Gamma \circ \pi_{12}^* \Gamma \Longrightarrow \pi_{13}^* \Gamma$ . For any path  $\Theta : \Xi \longrightarrow \Xi'$  in  $Y^{[3]}$ , the corresponding pasting diagram is

$$\begin{array}{ccc} \pi_1(\Xi) & \xrightarrow{\pi_1(\Theta)} & \pi_1(\Xi') \\ \pi_{12}(\Xi) \searrow \pi_{12}(\Theta) \swarrow \pi_{12}(\Xi') & & \\ \pi_{13}(\Xi) \xleftarrow{\pi_{13}(\Theta)} \pi_{13}(\Xi') & & \end{array} \quad (3.2.5)$$

Here we suppress writing the associators and the bracketing of the 1-morphisms. Using

this identity we have

$$\begin{array}{c}
 \begin{array}{ccccc}
 & \pi_1(\mu) & \xrightarrow{\pi_1(\Psi)} & \pi_1(\nu) & \\
 & \downarrow & \nearrow \pi_{12}(\Theta) & \downarrow & \\
 \Psi = \mu & \xleftarrow{\quad} c & \xrightarrow{\text{id}_c} & c & \xleftarrow{\quad} \nu \\
 & \downarrow & \nwarrow \pi_{23}(\Theta) & \downarrow & \\
 & \pi_2(\mu) & \xrightarrow{\pi_2(\Psi)} & \pi_2(\nu) & 
 \end{array}
 \end{array} \tag{3.2.6}$$

for  $c \in Y$  an arbitrary point with  $\pi(c) = p$  and  $\Theta := (\pi_1(\Psi), \text{id}_c, \pi_2(\Psi))$  which is only possible because  $p^\pi(\Psi) = \text{id}_{\text{id}_p}$ .

We can now assume that the 2-morphism  $\Xi : \alpha \Rightarrow \beta$  we started with contains no 2-morphism of type (1a) and by (3.2.6) only those 2-morphism  $\Theta = (\gamma_1, \gamma_2)$  for which either  $\gamma_1$  or  $\gamma_2$  is the identity path of the point  $c$ . If both  $\gamma_1$  and  $\gamma_2$  are identity paths, we can replace  $\Theta$  according to (3.2.3) by two 2-morphisms of type (1d). It is now a combinatorial task to kill all 2-morphisms which start or end on paths, in particular all 2-morphisms of type (2b). Then one kills all 2-morphisms of types (1d) and the remaining unifiers  $l_\beta^*$  and  $r_\beta^*$ . Finally, all associators  $a^*$  can be killed using their naturality with respect to 2-morphisms of type (1c). This proves (b).

To prove (c) we assume that  $\Xi' : \alpha \Rightarrow \beta$  is any 2-isomorphism with  $p^\pi(\Xi) = \text{id}_{\text{id}_p}$ . By (b) we can assume that  $\Xi'$  is composed only of 2-isomorphisms of type (1c). It is straightforward to see that two such compositions can be transformed into each other if six identities are satisfied: two *bubble* identities and four *fusion* identities. The two bubble identities are

$$\begin{array}{c}
 \begin{array}{ccc}
 & \pi_2(\Psi) & \\
 \nearrow & \Downarrow & \searrow \\
 \pi_1(\Psi) & \xrightarrow{\quad} & \pi_3(\Psi) \\
 \searrow & \Downarrow & \nearrow \\
 & \pi_2(\Psi) & 
 \end{array} = \text{id}_{\pi_{23}(\Psi) \circ \pi_{12}(\Psi)} \quad \text{and} \quad 
 \begin{array}{ccc}
 & \pi_{13}(\Psi) & \\
 \nearrow & \Downarrow & \searrow \\
 \pi_1(\Psi) & \xrightarrow{\quad} & \pi_3(\Xi) \\
 \searrow & \Downarrow & \nearrow \\
 & \pi_{13}(\Psi) & 
 \end{array} = \text{id}_{\pi_{13}(\Psi)}.
 \end{array}$$

They follow from the fact that the 2-morphisms of type (1c) are invertible. The first fusion

identity is identification (V1),

$$\begin{array}{ccc}
\pi_2(\Psi) & \xrightarrow{\pi_{23}(\Psi)} & \pi_3(\Psi) \\
\uparrow \pi_{12}(\Psi) & \nearrow \pi_{123}(\Psi) & \downarrow \pi_{34}(\Psi) \\
& \pi_{13}(\Psi) & \\
& \downarrow \pi_{134}(\Psi) & \\
\pi_1(\Psi) & \xrightarrow{\pi_{14}(\Psi)} & \pi_4(\Psi)
\end{array}
=
\begin{array}{ccc}
\pi_2(\Psi) & \xrightarrow{\pi_{23}(\Psi)} & \pi_3(\Psi) \\
& \searrow \pi_{234}(\Psi) & \downarrow \pi_{34}(\Psi) \\
& \pi_{24}(\Psi) & \\
& \downarrow \pi_{124}(\Psi) & \\
\pi_1(\Psi) & \xrightarrow{\pi_{14}(\Psi)} & \pi_4(\Psi)
\end{array}$$

The other three fusion identities are analogous identities for formal inverses  $\bar{\Psi}$  and mixtures of  $\Psi$  and  $\bar{\Psi}$ .  $\square$

Proof of Lemma 3.2.2 (b). Now let  $\gamma : x \longrightarrow y$  be a path in  $M$ , and let  $\tilde{x}, \tilde{y} \in Y$  be lifts of the endpoints, i.e.  $\pi(\tilde{x}) = x$  and  $\pi(\tilde{y}) = y$ . We compare the two lifts  $\tilde{\gamma}$  and  $\tilde{\gamma}'$  of  $\gamma$  in the following way. Let  $P \subseteq M$  be the set of points over whose fibre either  $\tilde{\gamma}_1$  or  $\tilde{\gamma}_2$  has a jump. The set  $P$  is finite and ordered by the orientation of the path  $\gamma$ , so that we may put  $P = \{p_0, \dots, p_n\}$  with  $p_0 = x$  and  $p_n = y$ . Now we write

$$\gamma = \gamma_n \circ \dots \circ \gamma_1$$

for paths  $\gamma_k : p_{k-1} \longrightarrow p_k$ . We also write  $\tilde{\gamma}$  as a composition of lifts  $\tilde{\gamma}_k : a_k \longrightarrow b_k$  of  $\gamma_k$  and (possibly multiple) jumps  $b_k \longrightarrow \alpha_{k+1}$  located over the points  $p_k$ ; analogously for  $\tilde{\gamma}'$ . This defines jumps  $\alpha_k := (a_k, a'_k)$  and  $\beta_k := (b_k, b'_k)$ . Now, over the paths  $\gamma_k$ , we take 2-isomorphisms

$$\begin{array}{ccc}
a_k & \xrightarrow{\tilde{\gamma}_k} & b_k \\
\alpha_k \downarrow & \searrow \Theta & \downarrow \beta_k \\
a'_k & \xrightarrow{\tilde{\gamma}'_k} & b'_k
\end{array} \quad (3.2.7)$$

with  $\Theta := (\tilde{\gamma}_k, \tilde{\gamma}'_k)$ . Over the points  $p_k$  we need 2-isomorphisms of the form

$$\begin{array}{ccc}
b_k = a_{k+1} & & \\
\beta_k \swarrow & \Downarrow & \searrow \alpha_{k+1} \\
b'_k & \xrightarrow{\quad} & a'_{k+1}
\end{array}, \quad
\begin{array}{ccc}
b_k & \xrightarrow{\quad} & a_{k+1} \\
\beta_k \swarrow & \searrow \alpha_{k+1} & \\
b'_k = a'_{k+1} & &
\end{array} \quad \text{or} \quad
\begin{array}{ccc}
b_k & \xrightarrow{\quad} & a_{k+1} \\
\beta_k \downarrow & \searrow & \downarrow \alpha_{k+1} \\
b'_k & \xrightarrow{\quad} & a'_{k+1}
\end{array} \quad (3.2.8)$$

the first whenever  $\tilde{\gamma}'$  has jumps over  $p_k$  and  $\tilde{\gamma}$  has not, the second whenever  $\tilde{\gamma}$  has jumps and  $\tilde{\gamma}'$  has not, and the third whenever both lifts have jumps. By Lemma 3.2.3 these

2-isomorphisms exist and are unique. Then, all of the four diagrams above can be put next to each other; this defines a 2-isomorphism  $\tilde{\gamma} \Longrightarrow \tilde{\gamma}'$ . We call the 2-morphism constructed like this the canonical 2-morphism.

It remains to show that every 2-morphism  $A : \tilde{\gamma} \Longrightarrow \tilde{\gamma}'$  with  $p^\pi(A) = \text{id}_\gamma$  is equal to this canonical 2-morphism. First, we kill all bigons contained in  $A$  by the argument given in the proof of Lemma 3.2.3. We consider two cases:

1.  $A$  contains no paths except those contained in  $\tilde{\gamma}$  or  $\tilde{\gamma}'$ . In this case  $A$  is already equal to the canonical 2-morphism. Namely, each of the pieces  $\tilde{\gamma}_k$  or  $\tilde{\gamma}'_k$  can only be target or source of a 2-morphism of type (1b). These are now necessarily the pieces (3.2.7). It remains to consider the 2-morphisms between the jumps. But these are by Lemma 3.2.3 equal to the pieces (3.2.8). This shows that  $A$  is the canonical 2-morphism.
2. There exists a path  $\gamma_0$  in  $\mathcal{P}_2^\pi(M)$  which is target or source of some 2-morphism contained in  $A$  but not contained in  $\tilde{\gamma}$  or  $\tilde{\gamma}'$ . In this case there exists a 1-morphism  $\tilde{\gamma}_o : \tilde{x} \longrightarrow \tilde{y}$  together with 2-morphisms  $A_1 : \tilde{\gamma} \Longrightarrow \tilde{\gamma}_o$  and  $A_2 : \tilde{\gamma}_o \Longrightarrow \tilde{\gamma}'$  such that  $A = A_2 \bullet A_1$ . By iteration, we can decompose  $A$  in a vertical composition of 2-morphisms which fall into case 1, i.e. into a vertical composition of canonical 2-morphisms.

It remains to conclude with the observation that the vertical composition  $A_2 \bullet A_1$  of two canonical 2-morphisms is again canonical.  $\square$

To construct the 2-functor  $s$  we fix an open cover  $\{U_i\}_{i \in I}$  of  $M$  together with smooth sections  $\sigma_i : U_i \longrightarrow Y$ , we fix choices of lifts  $s(p) \in Y$  for all points  $p \in M$ , and we fix for every path  $\gamma : x \longrightarrow y$  in  $M$  a 1-morphism  $s(\gamma) : s(x) \longrightarrow s(y)$  in  $\mathcal{P}_2^\pi(M)$ . Such lifts exist according to Lemma 3.2.2 (a). For the identity 1-morphisms  $\text{id}_x$  we choose the identity 1-morphisms  $\text{id}_{s(x)}^*$ . Moreover, we require  $s(\gamma^{-1}) = s(\gamma)^{-1}$ , meaning that  $s(\gamma)^{-1}$  is the reverse order composition of the inverses  $\tilde{\gamma}^{-1}$  of the involved paths  $\tilde{\gamma}$ , and of the inverses  $\bar{\alpha}$  of all involved jumps  $\alpha$ . These choices define  $s$  on objects and 1-morphisms.

Now let  $\Sigma : \gamma_1 \Longrightarrow \gamma_2$  be a bigon in  $M$ . Its image  $\Sigma([0,1]^2) \subseteq M$  is compact and hence covered by open sets indexed by a *finite* subset  $J \subseteq I$ . We choose a decomposition of  $\Sigma$  in a vertical and horizontal composition of bigons  $\{\Sigma_j\}_{j \in J}$  such that  $\Sigma_j([0,1]^2) \subseteq U_j$ . Then we define  $s(\Sigma)$  to be composed from 2-morphisms  $s(\Sigma_j)$  in the same way as  $\Sigma$  was composed from the  $\Sigma_j$ . It thus remains to define the 2-functor  $s$  on bigons  $\Sigma$  which are

contained in an open set  $U$  which has a section  $\sigma : U \rightarrow Y$ . We define for such a bigon

$$s : \begin{array}{c} \begin{array}{ccc} & \gamma_1 & \\ x & \Downarrow \Sigma & y \\ & \gamma_2 & \end{array} \end{array} \mapsto \begin{array}{c} \begin{array}{ccccc} & s(\gamma_1) & & & \\ & \Downarrow \sigma(\gamma_1) & & & \\ s(x) & \longrightarrow & \sigma(x) & \xrightarrow{\sigma(\Sigma)} & \sigma(y) & \longrightarrow & s(y) \\ & \Downarrow \sigma(\gamma_2) & & & \\ & s(\gamma_2) & & & \end{array} \end{array}$$

where the unlabelled 1-morphisms are the obvious jumps, and the unlabelled 2-morphisms are the unique 2-isomorphisms from Lemma 3.2.2 (b).

The 2-functor  $s : \mathcal{P}_2(M) \rightarrow \mathcal{P}_2^\pi(M)$  whose structure we have defined above is not strict. While its unitor is trivial because we have by definition  $s(\text{id}_x) = \text{id}_{s(x)}^*$ , its compositor  $c_{\gamma_1, \gamma_2} : s(\gamma_2) \circ s(\gamma_1) \Rightarrow s(\gamma_2 \circ \gamma_1)$  is defined to be the unique 2-isomorphism from Lemma 3.2.2 (b). All axioms for the 2-functor  $s$  follow from the uniqueness of these 2-isomorphisms.

For later purpose, we note the following consequence of the definitions.

**Lemma 3.2.4.** *If  $\gamma : x \rightarrow y$  is a path in  $M$ , then the compositor  $c_{\gamma, \gamma^{-1}}$  of  $s$  is a composition of 2-morphisms of types (2a) and (2b), type (1d), and those 2-morphisms  $\Xi \in Y^{[3]}$  of type (1c) that are in the image of  $\Delta_{121} : Y^{[2]} \rightarrow Y^{[3]} : (a, b) \mapsto (a, b, a)$ .*

Now we can proceed with the remaining proof of the main result of this section.

**Proof of Proposition 3.2.1.** By construction we have  $p^\pi \circ s = \text{id}_{\mathcal{P}_2(M)}$ . It remains to construct a pseudonatural equivalence

$$\zeta : s \circ p^\pi \rightarrow \text{id}_{\mathcal{P}_2^\pi(M)}.$$

We define  $\zeta$  on both basic 1-morphisms. Its component at a path is

$$\zeta : \begin{array}{ccc} a & \xrightarrow{\gamma} & b \end{array} \mapsto \begin{array}{ccc} s(\pi(a)) & \xrightarrow{s(\pi_*\gamma)} & s(\pi(b)) \\ \downarrow & \nearrow & \downarrow \\ a & \xrightarrow{\gamma} & b \end{array}$$

where the unlabelled 1-morphisms are again the obvious jumps, and the 2-isomorphism is the unique one. If  $s(\pi_*\gamma)$  happens to be just a path, this 2-isomorphism is just of type (1b). The component of  $\zeta$  at a jump is

$$\zeta \quad : \quad \pi_1(\alpha) \xrightarrow{\alpha} \pi_2(\alpha) \quad \mapsto \quad \begin{array}{ccc} & s(x) & \\ \nearrow & \Downarrow & \searrow \\ \pi_1(\alpha) & \xrightarrow{\quad} & \pi_2(\alpha) \end{array}$$

with  $x := \pi(\pi_1(\alpha)) = \pi(\pi_2(\alpha))$ ; this is just one 2-isomorphism of type (1c). For some general 1-morphism,  $\zeta$  puts the 2-isomorphisms above next to each other; this way axiom (T1) is automatically satisfied. Axiom (T2) follows again from the uniqueness of the 2-morphisms we have used.

In order to show that  $\zeta$  is invertible we need to find another pseudonatural transformation  $\xi : \text{id}_{\mathcal{P}_2^\pi(M)} \longrightarrow s \circ p^\pi$  together with invertible modifications  $i_\zeta : \xi \circ \zeta \Longrightarrow \text{id}_{s \circ p^\pi}$  and  $j_\zeta : \text{id}_{\text{id}_{\mathcal{P}_2^\pi(M)}} \Longrightarrow \zeta \circ \xi$  that satisfy the zigzag identities. The pseudonatural transformation  $\xi$  can be defined in the same way as  $\zeta$  just by turning the diagrams upside down, using the formal inverses. The modifications  $i_\zeta$  and  $j_\zeta$  assign to a point  $a \in Y$  the 2-isomorphisms

$$\begin{array}{ccc} \zeta(a) & \xrightarrow{\quad} & a \\ \downarrow & \Delta(s(\pi(a))) & \downarrow \\ s(\pi(a)) & \xrightarrow{\quad} & s(\pi(a)) \end{array} \quad \text{and} \quad \begin{array}{ccc} & \text{id}_a^* & \\ \nearrow & \Downarrow & \searrow \\ a & \xrightarrow{\quad} & a \\ \downarrow & \Delta(a) & \downarrow \\ \zeta(a) & \xrightarrow{\quad} & s(\pi(a)) \end{array}$$

that combine 2-isomorphisms of type (1c) and (1d). The zigzag identities are satisfied due to the uniqueness of 2-isomorphisms we have used.  $\square$

**Corollary 3.2.5.** *The section 2-functor  $s : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2^\pi(M)$  is independent (up to pseudonatural equivalence) of all choices, namely the choice of lifts of points and 1-morphisms, the choice of the open cover, and the choice of local sections.*

This follows from the fact that two weak inverses of a 1-morphism in a 2-category are automatically 2-isomorphic.



### 3.3 Pairing with Descent Data

In this section we relate the codescent 2-groupoid  $\mathcal{P}_2^\pi(M)$  to the descent 2-category  $\mathfrak{Des}_\pi^2(i)$  defined in Section 2.2 in terms of a strict 2-functor

$$R : \mathfrak{Des}_\pi^2(i) \longrightarrow \text{Funct}(\mathcal{P}_2^\pi(M), T).$$

This 2-functor expresses that the 2-groupoid  $\mathcal{P}_2^\pi(M)$  is “ $T$ -dual” to the descent 2-category; this justifies the term *codescent* 2-groupoid.

The 2-functor  $R$  labels the structure of the codescent 2-groupoid by descent data in a natural way. To start with, let  $(\text{triv}, g, \psi, f)$  be a descent object. Its image under  $R$  is a 2-functor

$$R_{(\text{triv}, g, \psi, f)} : \mathcal{P}_2^\pi(M) \longrightarrow T$$

defined as follows. To an object  $a \in Y$  it assigns the object  $\text{triv}_i(a)$  in  $T$ . On basic 1-morphisms it is defined by the following assignments:

$$\begin{aligned} a \xrightarrow{\gamma} a' &\longmapsto \text{triv}_i(a) \xrightarrow{\text{triv}_i(\gamma)} \text{triv}_i(a') \\ \pi_1(\alpha) \xrightarrow{\alpha} \pi_2(\alpha) &\longmapsto \pi_1^* \text{triv}_i(\alpha) \xrightarrow{g(\alpha)} \pi_2^* \text{triv}_i(\alpha). \end{aligned}$$

To a formal composition of basic 1-morphisms it assigns the composition of the respective images, and to the formal identity  $\text{id}_a^*$  at a point  $a \in Y$  it assigns  $\text{id}_{\text{triv}_i(a)}$ . On the basic 2-morphisms of essential types (1a) to (1d) it is defined by the following assignments:

$$\begin{aligned} \begin{array}{ccc} & \gamma_1 & \\ a & \begin{array}{c} \downarrow \text{ } \downarrow \end{array} & b \\ & \gamma_2 & \end{array} &\longmapsto \begin{array}{ccc} & \text{triv}_i(\gamma_1) & \\ \text{triv}_i(a) & \begin{array}{c} \downarrow \text{ } \downarrow \end{array} & \text{triv}_i(b) \\ & \text{triv}_i(\gamma_2) & \end{array} \\ \\ \begin{array}{ccc} \pi_1(\alpha) & \xrightarrow{\pi_1(\Theta)} & \pi_1(\alpha') \\ \alpha \downarrow & \swarrow \Theta & \downarrow \alpha' \\ \pi_2(\alpha) & \xrightarrow{\pi_2(\Theta)} & \pi_2(\alpha') \end{array} &\longmapsto \begin{array}{ccc} \pi_1^* \text{triv}_i(\alpha) & \xrightarrow{\pi_1^* \text{triv}_i(\Theta)} & \pi_1^* \text{triv}_i(\alpha') \\ g(\alpha) \downarrow & \swarrow g(\Theta) & \downarrow g(\alpha') \\ \pi_2^* \text{triv}_i(\alpha) & \xrightarrow{\pi_2^* \text{triv}_i(\Theta)} & \pi_2^* \text{triv}_i(\alpha') \end{array} \end{aligned}$$

$$\begin{array}{ccc}
\begin{array}{ccc}
& \pi_2(\Xi) & \\
\pi_{12}(\Xi) \nearrow & \Downarrow & \searrow \pi_{23}^*(\Xi) \\
\pi_1(\Xi) & \xrightarrow{\pi_{13}(\Xi)} & \pi_3(\Xi)
\end{array} & \mapsto & \begin{array}{ccc}
& \pi_2^* \text{triv}_i(\Xi) & \\
\pi_{12}^* g(\Xi) \nearrow & \Downarrow f(\Xi) & \searrow \pi_{23}^* g^*(\Xi) \\
\pi_1^* \text{triv}_i(\Xi) & \xrightarrow{\pi_{13}^* g(\Xi)} & \pi_3^* \text{triv}_i(\Xi)
\end{array} \\
\\
\text{id}_a^* \xRightarrow{\Delta_a} \Delta(a) & \mapsto & \text{id}_{\text{triv}_i(a)} \xRightarrow{\psi(a)} \Delta^* g(a).
\end{array}$$

To the basic 2-morphisms of technical type (2a) it assigns associators and unifiers of the 2-category  $T$ . To those of type (2b) it assigns unitors and compositors of the 2-functor  $i$ , i.e.

$$\begin{array}{ccc}
\text{id}_a \xRightarrow{u_a^*} \text{id}_a^* & \mapsto & \text{triv}_i(\text{id}_a) \xRightarrow{u_{\text{triv}(a)}^i} \text{id}_{\text{triv}_i(a)} \\
\gamma_2 * \gamma_1 \xRightarrow{c_{\gamma_1, \gamma_2}^*} \gamma_2 \circ \gamma_1 & \mapsto & \text{triv}_i(\gamma_2) \circ \text{triv}_i(\gamma_1) \xRightarrow{c_{\text{triv}(\gamma_1), \text{triv}(\gamma_2)}^i} \text{triv}_i(\gamma_2 \circ \gamma_1).
\end{array}$$

Finally, some formal horizontal and vertical composition of 2-morphisms is assigned to the composition of the images of the respective basic 2-morphisms, the formal horizontal composition replaced by the horizontal composition  $\circ$  of  $T$ , and the formal vertical composition replaced by the vertical composition  $\bullet$  of  $T$ .

By construction, all these assignments are well-defined under the identifications we have declared under the 2-morphisms of  $\mathcal{P}_2^\pi(M)$ :

- They are well-defined under the identifications (I) due to the axioms of the 2-category  $T$ .
- They are well-defined under identifications (II) due to the axioms of the 2-functors  $\text{triv}$  and  $i$ .
- They are well-defined under identifications (III) due to the axioms of the pseudonatural transformation  $g$ .
- They are well-defined under identifications (IV) due to the axioms of the modifications  $\psi$  and  $f$ .

- They are well-defined under the identifications (V) because these are explicitly assumed in the definition of descent objects, see diagrams (2.2.1) and (2.2.2).

We have now defined the 2-functor  $R_{(\text{triv}, g, \psi, f)}$  on descent objects, 1-morphisms and 2-morphisms. Since for all points  $a \in Y$

$$R_{(\text{triv}, g, \psi, f)}(\text{id}_a^*) = \text{id}_{\text{triv}_i(a)} = \text{id}_{R_{(\text{triv}, g, \psi, f)}(a)},$$

it has a trivial unitor. Furthermore,

$$R_{(\text{triv}, g, \psi, f)}(\gamma) \circ R_{(\text{triv}, g, \psi, f)}(\beta) = R_{(\text{triv}, g, \psi, f)}(\gamma * \beta)$$

for all composable 1-morphisms  $\beta$  and  $\gamma$  of any type, so that it also has a trivial compositor. Hence, the 2-functor  $R_{(\text{triv}, g, \psi, f)}$  is strict, and it is straightforward to see that the remaining axioms (F1) and (F2) are satisfied.

So far we have introduced a 2-functor associated to each descent object. Let us now introduce a pseudonatural transformation

$$R_{(h, \epsilon)} : R_{(\text{triv}, g, \psi, f)} \longrightarrow R_{(\text{triv}', g', \psi', f')}$$

associated to a descent 1-morphism

$$(h, \epsilon) : (\text{triv}, g, \psi, f) \longrightarrow (\text{triv}', g', \psi', f').$$

Its definition is as natural as the one of the 2-functor given before. Its component at an object  $a \in Y$  is the 1-morphism

$$h(a) : \text{triv}_i(a) \longrightarrow \text{triv}'_i(a).$$

Its components at basic 1-morphisms are given by the following assignments:

$$\begin{array}{ccc}
a \xrightarrow{\gamma} a' & \mapsto & 
\begin{array}{ccc}
\text{triv}_i(a) & \xrightarrow{\text{triv}_i(\gamma)} & \text{triv}_i(a') \\
\downarrow h(a) & \swarrow h(\gamma) & \downarrow h(a') \\
\text{triv}'_i(a) & \xrightarrow{\text{triv}'_i(\gamma)} & \text{triv}'_i(a')
\end{array} \\
\\
\pi_1(\alpha) \xrightarrow{\alpha} \pi_2(\alpha) & \mapsto & 
\begin{array}{ccc}
\pi_1^* \text{triv}_i(\alpha) & \xrightarrow{g(\alpha)} & \pi_2^* \text{triv}_i(\alpha) \\
\downarrow \pi_1^* h(\alpha) & \swarrow \epsilon(\alpha) & \downarrow \pi_2^* h(\alpha) \\
\pi_1^* \text{triv}'_i(\alpha) & \xrightarrow{g'(\alpha)} & \pi_2^* \text{triv}'_i(\alpha)
\end{array}
\end{array}$$

For compositions of 1-morphisms,  $R_{(h,\epsilon)}$  puts the diagrams for the involved basic 1-morphisms next to each other. For example, to a composition  $\gamma * \alpha$  of a jump  $\alpha = (x, y)$  with a path  $\gamma : y \longrightarrow z$  it assigns the 2-isomorphism

$$h(z) \circ (\text{triv}_i(\gamma) \circ g(\alpha)) \Longrightarrow (\text{triv}'_i(\gamma) \circ g(\alpha)) \circ h(x)$$

which is (up to the obvious associators) obtained by first applying  $h(\gamma)$  and then  $\epsilon(\alpha)$ . This way, axiom (T1) for the pseudonatural transformation  $R_{(h,\epsilon)}$ , namely the compatibility with the composition of 1-morphisms, is automatically satisfied. It remains to prove the following.

**Lemma 3.3.1.** *The assignments  $R_{(h,\epsilon)}$  are compatible with the 2-morphisms of the codescent 2-groupoid in the sense of axiom (T2).*

*Proof.* We check the compatibility separately for each basic 2-morphism. For the essential 2-morphisms it comes from the following properties of the descent 1-morphism  $(h, \epsilon)$ :

- For type (1a) it comes from axiom (T2) for the pseudonatural transformation  $h$ .
- For type (1b) it comes from the axiom for the modification  $\epsilon$  and from axiom (T2) for the pseudonatural transformation  $h$ .
- For types (1c) and (1d) it comes from the conditions (2.2.3) and (2.2.4) on the descent 1-morphism  $(h, \epsilon)$ .

For the technical 2-morphisms it comes from properties of the 2-category  $T$  and the one of the 2-functor  $i$ : for type (2a) it is satisfied because the associators and unifiers of  $T$  are natural, and for type (2b) it is satisfied because the compositors and unitors of  $i$  are natural.  $\square$

We have now described a 2-functor associated to each descent object and a pseudonatural transformation associated to each descent 1-morphism. Now let  $(\text{triv}, g, \psi, f)$  and  $(\text{triv}', g', \psi', f')$  be descent objects and let  $(h_1, \epsilon_1)$  and  $(h_2, \epsilon_2)$  be two descent 1-morphisms between these. For a descent 2-morphism  $E : (h_1, \epsilon_1) \Longrightarrow (h_2, \epsilon_2)$  we introduce now a modification

$$R_E : R_{(h_1, \epsilon_1)} \Longrightarrow R_{(h_2, \epsilon_2)}.$$

Its component at an object  $a \in Y$  is the 2-morphism  $E(a) : h_1(a) \Longrightarrow h_2(a)$ . The axiom for  $R_E$ , the compatibility with 1-morphisms, is satisfied for paths because  $E$  is a modification, and for jumps because of the diagram (2.2.5) in the definition of descent 2-morphisms.

It is now straightforward to see the following statement.

**Proposition 3.3.2.** *The assignments defined above furnish a strict 2-functor*

$$R : \mathfrak{Des}_\pi^2(i) \longrightarrow \text{Funct}(\mathcal{P}_2^\pi(M), T).$$

The 2-functor  $R$  “represents” the descent 2-category in a 2-category of 2-functors; in fact in a faithful way. We recall from Section 2.2 that there is a 2-functor  $V : \mathfrak{Des}_\pi^2(i) \longrightarrow \text{Funct}(\mathcal{P}_2(Y), T)$  which is also a representation of the same kind (but not faithful). The relation between these two representations is the following observation.

**Lemma 3.3.3.**  $\iota^* \circ R = V$ , where  $\iota^*$  denotes the composition with  $\iota : \mathcal{P}_2(Y) \longrightarrow \mathcal{P}_2^\pi(M)$ .

From this point of view, the codescent 2-groupoid enlarges the path 2-groupoid  $\mathcal{P}_2(Y)$  by additional 1-morphisms (the jumps) and additional 2-morphisms in such a way that it carries a faithful representation of the descent 2-category.

Now we put the two main aspects of the codescent 2-groupoid together, namely the representation 2-functor  $R$ , and its equivalence with the path 2-groupoid in terms of the section 2-functor  $s$ : the *reconstruction 2-functor*

$$\text{Rec}_\pi : \mathfrak{Des}_\pi^2(i) \longrightarrow \text{Triv}_\pi^2(i)$$

is defined to be the composition

$$\mathfrak{Des}_\pi^2(i) \xrightarrow{R} \text{Funct}(\mathcal{P}_2^\pi(M), T) \xrightarrow{s^*} \text{Funct}(\mathcal{P}_2(M), T) .$$

Here,  $s^*$  is the composition with  $s$ . According to Corollary 3.2.5, the reconstruction 2-functor is (up to pseudonatural equivalence) canonically attached to the surjective submersion  $\pi : Y \longrightarrow M$  and the 2-functor  $i : \text{Gr} \longrightarrow T$ .

In order to show that the reconstruction ends in the sub-2-category  $\text{Triv}_\pi^2(i)$  of  $\text{Funct}(\mathcal{P}_2(M), T)$  it remains to equip, for each descent object  $(\text{triv}, g, \psi, f)$ , the reconstructed 2-functor

$$F := R_{(\text{triv}, g, \psi, f)} \circ s$$

with a  $\pi$ -local  $i$ -trivialization  $(\text{triv}, t)$ . Clearly, we take the given 2-functor  $\text{triv}$  as the first ingredient and are left with the construction of a pseudonatural equivalence

$$t : \pi^* F \longrightarrow \text{triv}_i. \tag{3.3.1}$$

This equivalence is simply defined by

$$\begin{array}{ccccc}
\mathcal{P}_2(Y) & \xrightarrow{\pi_*} & \mathcal{P}_2(M) & & \\
\downarrow \text{triv} & \searrow \iota & \nearrow p^\pi & \nearrow \zeta & \downarrow s \\
& & \mathcal{P}_2^\pi(M) & & \\
& & \searrow \text{id} & \searrow & \\
& & \mathcal{P}_2^\pi(M) & & \\
& & \downarrow R_{(\text{triv}, g, \psi, f)} & & \\
\text{Gr} & \xrightarrow{i} & T & & 
\end{array}$$

where  $\zeta$  is the pseudonatural equivalence from Section 3.2. The triangle on the top of the latter diagram is equation (3.2.1), and the remaining subdiagram expresses the equation

$$\iota^* R_{(\text{triv}, g, \psi, f)} = \text{triv}_i$$

which follows from Lemma 3.3.3.

We conclude with a lemma about the reconstruction of 2-functors from normalized descent objects.

**Lemma 3.3.4.** *Suppose  $T$  is a strict 2-category,  $i : \text{Gr} \rightarrow T$  is a strict 2-functor, and  $(\text{triv}, g, \psi, f)$  is a normalized descent object. Then, the reconstructed 2-functor*

$$R_{(\text{triv}, g, \psi, f)} \circ s : \mathcal{P}_2(M) \rightarrow T$$

*is normalized in the sense explained in Appendix A.*

*Proof.* We write  $R := R_{(\text{triv}, g, \psi, f)}$  and  $F := R \circ s$  for abbreviation. Let  $x \in M$ . Since  $R$  is strict and  $s$  has trivial unitor,  $F$  has a trivial unitor:

$$u_x^F = u_{s(x)}^R \bullet R(u_x^s) = R(\text{id}_{\text{id}_{s(x)}^*}) = \text{id}_{R(\text{id}_{s(x)}^*)} = \text{id}_{\text{id}_{\text{triv}_i(s(x))}}.$$

Let  $\gamma : x \rightarrow y$  be a path. Since  $R$  is strict, we have

$$c_{\gamma, \gamma^{-1}}^F = R(c_{\gamma, \gamma^{-1}}^s).$$

By Lemma 3.2.4 the compositors  $c_{\gamma, \gamma^{-1}}^s$  consists of 2-morphisms of types (2a), (2b), (1c) and (1d). Since  $T$  and  $i$  are strict, all 2-morphisms of technical type (2a) and (2b) are sent by  $R$  to identities. The 2-morphisms of types (1c) and (1d) are sent by  $R$  to the components of  $\psi$  and  $f$ , and these are identities since  $(\text{triv}, g, \psi, f)$  is a normalized descent object.  $\square$

## 4 Local - Global Equivalence

In this section we prove the main theorem of this article, namely that extraction and reconstruction establish an equivalence between locally defined descent data and globally defined 2-functors.

### 4.1 Equivalence for a Fixed Cover

Let  $i : \text{Gr} \rightarrow T$  be a 2-functor from a strict 2-groupoid  $\text{Gr}$  to a 2-category  $T$ , and let  $\pi : Y \rightarrow M$  be a surjective submersion.

**Proposition 4.1.1.** *The 2-functor*

$$\text{Ex}_\pi : \text{Triv}_\pi^2(i) \rightarrow \mathfrak{Des}_\pi^2(i)$$

*is an equivalence of 2-categories.*

For the proof we shall choose a section 2-functor  $s : \mathcal{P}_2(M) \rightarrow \mathcal{P}_2^\pi(M)$  defining the reconstruction 2-functor  $\text{Rec}_\pi$ . We show that  $\text{Ex}_\pi$  and  $\text{Rec}_\pi$  form an equivalence of 2-categories. This is done in the following two lemmata.

**Lemma 4.1.2.** *There is a pseudonatural equivalence  $\text{Ex}_\pi \circ \text{Rec}_\pi \cong \text{id}_{\mathfrak{Des}_\pi^2(i)}$ .*

Proof. Given a descent object  $(\text{triv}, g, \psi, f)$  let us pass to the reconstructed 2-functor and extract its descent data  $(\text{triv}', g', \psi', f')$ . We find immediately  $\text{triv}' = \text{triv}$ . Furthermore, the pseudonatural transformation  $g'$  has the components

$$g' : \alpha \xrightarrow{\Theta} \alpha' \mapsto \begin{array}{ccc} \pi_1^* \text{triv}_i(\alpha) & \xrightarrow{\pi_1^* \text{triv}_i(\Theta)} & \pi_1^* \text{triv}_i(\alpha') \\ \swarrow & \nearrow & \swarrow \\ c_\alpha & \xleftarrow{f(\Xi_\alpha)^{-1}} df & \xrightarrow{g(\Theta)} & \xrightarrow{f(\Xi_{\alpha'})} c_{\alpha'} \\ \searrow & \nwarrow & \searrow \\ \pi_2^* \text{triv}_i(\alpha) & \xrightarrow{\pi_1^* \text{triv}_i(\Theta)} & \pi_2^* \text{triv}_i(\alpha') \end{array}$$

where we have introduced an object  $c_\alpha := \text{triv}_i(s(p))$  where  $p = \pi(\pi_1(\alpha)) = \pi(\pi_2(\alpha))$  and a 2-morphism  $\Xi_\alpha := (\pi_1(\alpha), s(p), \pi_2(\alpha))$ . It is useful to notice that this means that  $f$  is a

modification  $f : g' \Rightarrow g$ . The modification  $\psi'$  has the component

$$\begin{array}{ccc}
 & \text{id}_{\text{triv}_i(a)} & \\
 & \downarrow \psi(a) & \\
 \text{triv}_i(a) & \xrightarrow{\quad} & \text{triv}_i(a) \\
 & \downarrow f(\Psi) & \\
 & c_{\Delta(a)} & 
 \end{array}$$

at a point  $a \in Y$ . Finally, the modification  $f'$  has the component

$$\begin{array}{ccccc}
 & & \pi_2^* \text{triv}_i(\Xi) & & \\
 & \nearrow \pi_{12}^* g'(\Xi) & \downarrow f(\Psi) & \nwarrow \pi_{23}^* g'(\Xi) & \\
 & c_{\Xi} & \xrightarrow{\quad} & c_{\Xi} & \\
 & \nwarrow \pi_1^* \text{triv}_i(\Xi) & \downarrow \psi(p)^{-1} & \nearrow \pi_3^* \text{triv}_i(\Xi) & \\
 & c_{\Xi} & \xrightarrow{\quad} & c_{\Xi} & \\
 & \nwarrow \pi_{13}^* g'(\Xi) & \downarrow \text{id}_{c_{\Xi}} & \nearrow & 
 \end{array}$$

at a point  $\Xi \in Y^{[3]}$ , where we have introduced the 2-morphism  $\Psi := (c_{\Xi}, \pi_2^* \text{triv}_i(\Xi), c_{\Xi})$ , and  $p$  is again the projection of  $\Xi$  to  $M$ . Now it is straightforward to construct a descent 1-morphism

$$\rho_{(\text{triv}, g, \psi, f)} : (\text{triv}, g', \psi', f') \longrightarrow (\text{triv}, g, \psi, f)$$

which consists of the identity pseudonatural transformation  $h := \text{id}_{\text{triv}}$  and of a modification  $\epsilon : \pi_2^* h \circ g' \Rightarrow g \circ \pi_1^* h$  induced from the modification  $f : g' \Rightarrow g$  and the left and right unifiers. This descent 1-morphism is the component of the pseudonatural equivalence  $\rho : \text{Ex}_{\pi} \circ \text{Rec}_{\pi} \longrightarrow \text{id}$  we have to construct, at the object  $(\text{triv}, g, \psi, f)$ .

Let us now define the component of  $\rho$  at a descent 1-morphism

$$(h, \epsilon) : (\text{triv}_1, g_1, \psi_1, f_1) \longrightarrow (\text{triv}_2, g_2, \psi_2, f_2).$$

It is useful to introduce a modification  $\tilde{\epsilon} : \bar{g}_2 \circ \pi_2^* h \circ g_1 \Rightarrow \pi_1^* h$  where  $\bar{g}_2$  is the pullback of  $g_2$  along the map  $Y^{[2]} \longrightarrow Y^{[2]}$  that exchanges the components. It is defined as the following composition of modifications:

$$\begin{array}{c}
 \bar{g}_2 \circ \pi_2^* h \circ g_1 \xrightarrow{\text{id} \circ \epsilon} \bar{g}_2 \circ g_2 \circ \pi_1^* h \\
 \downarrow \Delta_{121}^* f_2 \circ \text{id} \\
 \pi_1^* \Delta^* g \circ \pi_1^* h \xrightarrow{\pi_1^* \psi_2^{-1} \circ \text{id}} \pi_1^* \text{id} \circ \pi_1^* h \xrightarrow{l_{\pi_1^* h}} \pi_1^* h
 \end{array}$$



Now, if we reconstruct and extract local data  $(h', \epsilon')$ , the pseudonatural transformation  $h'$  has the components

$$\begin{array}{c}
 \begin{array}{ccc}
 & i(\text{triv}_1(a)) & \xrightarrow{i(\text{triv}_1(\gamma))} i(\text{triv}_2(b)) \\
 & \swarrow & \searrow \\
 i(\pi_2^* \text{triv}_1(\alpha)) & & i(\pi_2^* \text{triv}_1(\beta)) \\
 \downarrow \pi_2^* h(\alpha) & \xleftarrow{\tilde{\epsilon}(\alpha)^{-1}} h(a) & \xrightarrow{h(\gamma)} h(b) \xleftarrow{\tilde{\epsilon}(\beta)} \downarrow \pi_2^* h(\beta) \\
 i(\pi_2^* \text{triv}_2(\alpha))^2 & & i(\pi_2^* \text{triv}_2(\beta)) \\
 & \searrow & \swarrow \\
 & i(\text{triv}_2(a)) & \xrightarrow{i(\text{triv}_2(\gamma))} i(\text{triv}_2(b))
 \end{array}
 \end{array}
 \quad h' : a \xrightarrow{\gamma} b \quad \mapsto$$

with  $\alpha := (a, s(\pi(a)))$  and  $\beta = (b, s(\pi(b)))$ . Like above we observe that  $\tilde{\epsilon}$  is hence a modification  $\tilde{\epsilon} : h' \Rightarrow h$ . Now, the component  $\rho_{(h, \epsilon)}$  we have to define is a descent 2-morphism

$$\begin{array}{ccc}
 (\text{triv}'_1, g', \psi', f') & \xrightarrow{(h', \epsilon')} & (\text{triv}'_2, g'_2, \psi'_2, f'_2) \\
 \downarrow \rho_{(\text{triv}_1, g_1, \psi_1, f_1)} & \swarrow \rho_{(h, \epsilon)} & \downarrow \rho_{(\text{triv}_2, g_2, \psi_2, f_2)} \\
 (\text{triv}_1, g_1, \psi_1, f_1) & \xrightarrow{(h, \epsilon)} & (\text{triv}_2, g_2, \psi_2, f_2),
 \end{array}$$

this is just a modification  $\text{id} \circ h' \Rightarrow h \circ \text{id}$  since the vertical arrows are the identity pseudonatural transformations. We define  $\rho_{(h, \epsilon)}$  from  $\tilde{\epsilon}$  and right and left unifiers in the obvious way. It is straightforward to see that this defines indeed a descent 2-morphism. Finally, we observe that the definitions  $\rho_{(\text{triv}, g, \psi, f)}$  and  $\rho_{(h, \epsilon)}$  furnish a pseudonatural equivalence as required.  $\square$

The second part of the proof of Proposition 4.1.1 is the following lemma.

**Lemma 4.1.3.** *There is a pseudonatural equivalence  $\text{id}_{\text{Triv}_\pi^2(i)} \cong \text{Rec}_\pi \circ \text{Ex}_\pi$ .*

Proof. For a 2-functor  $F : \mathcal{P}_2(X) \rightarrow T$  and a  $\pi$ -local  $i$ -trivialization  $(\text{triv}, t)$ , let  $(\text{triv}, g, \psi, f)$  be the associated descent data. We find a pseudonatural transformation

$$\eta_F : F \rightarrow s^* R_{(\text{triv}, g, \psi, f)}$$

in the following way. Its component at a point  $x \in X$  is the 1-morphism  $t(s(x)) : F(x) \rightarrow \text{triv}_i(s(x))$  in  $T$ . To define its component at a path  $\gamma : x \rightarrow y$

we recall that  $s(\gamma)$  is a composition of paths  $\gamma_i : a_i \longrightarrow b_i$  and jumps  $\alpha_i$ , so that we can compose  $\eta_F(\gamma)$  from the pieces

$$\begin{array}{ccc}
\pi^* F(a_i) & \xrightarrow{\pi^* F(\gamma_i)} & \pi^* F(b_i) \\
\eta_F(a_i) \downarrow & \swarrow t(\gamma_i) & \downarrow \eta_F(b_i) \\
\text{triv}_i(a_i) & \xrightarrow{\text{triv}_i(\gamma)} & \text{triv}_i(b_i)
\end{array}
\quad \text{and} \quad
\begin{array}{ccccc}
& & F(p) & & \\
& \swarrow \eta_F(\pi_1(\alpha)) & \downarrow \text{id} & \searrow \eta_F(\pi_2(\alpha)) & \\
\pi_1^* \text{triv}_i(\alpha) & \xrightarrow{\pi_1^* t(\alpha)} & F(p) & \xrightarrow{\pi_2^* t(\alpha)} & \pi_2^* \text{triv}_i(\alpha) \\
& \searrow i_t^{-1} & & \swarrow i_t & \\
& & g(\alpha) & & 
\end{array}$$

where  $i_t : \bar{t} \circ t \Longrightarrow \text{id}$  is the modification chosen to extract descent data. This defines the pseudonatural transformation  $\eta_F$  associated to a 2-functor  $F$ .

Now let  $A : F_1 \longrightarrow F_2$  be a pseudonatural transformation between two 2-functors with local trivializations  $(\text{triv}_1, t_1)$  and  $(\text{triv}_2, t_2)$ . Let  $(h, \epsilon)$  the associated descent 1-morphism. It is now straightforward to see that

$$\eta_A := i_{t_1}^{-1} : \eta_{F_2} \circ A \Longrightarrow s^* R_{(h, \epsilon)} \circ \eta_{F_1}$$

defines a modification in such a way that both definitions together yield a pseudonatural transformation  $\eta : \text{id}_{\text{Triv}_{\pi}^2(i)} \longrightarrow \text{Rec}_{\pi} \circ \text{Ex}_{\pi}$ . It is clear that  $\eta$  is even a pseudonatural equivalence.  $\square$

## 4.2 Equivalence in the Direct Limit

As mentioned in Section 2.1, path 2-groupoids come with 2-functors  $f_* : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2(N)$  associated to smooth maps  $f : M \longrightarrow N$ . In turn, these define 2-functors

$$f^* : \text{Funct}(\mathcal{P}_2(N), T) \longrightarrow \text{Funct}(\mathcal{P}_2(M), T).$$

The compatibility (2.1.1) of the 2-functors  $f_*$  with the composition of smooth maps show that

$$(g \circ f)^* = f^* \circ g^* \tag{4.2.1}$$

for  $g : N \longrightarrow O$  another smooth map.

Now let  $\pi_1 : Y_1 \longrightarrow M$  and  $\pi_2 : Y_2 \longrightarrow M$  be surjective submersions and let  $\xi : Y_1 \longrightarrow Y_2$  be a smooth map such that  $\pi_2 \circ \xi = \pi_1$ . We call  $\xi$  a *refinement* of  $\pi_2$ . Equation (4.2.1) implies that we obtain induced “restriction” 2-functors

$$\text{res}_{\xi} : \text{Triv}_{\pi_2}^2(i) \longrightarrow \text{Triv}_{\pi_1}^2(i) \quad \text{and} \quad \text{res}_{\xi} : \mathfrak{Des}_{\pi_2}^2(i) \longrightarrow \mathfrak{Des}_{\pi_1}^2(i), \tag{4.2.2}$$

and that these 2-functors themselves satisfy the compatibility condition (4.2.1), with respect to iterated refinements of surjective submersions.

In general, suppose that  $S$  is a family of 2-categories parameterized by surjective submersions over a smooth manifold  $M$ . That is, if  $\pi : Y \rightarrow M$  is a surjective submersion, then  $S(\pi)$  is a 2-category. Suppose further that  $F$  is a family of “refinement” 2-functors parameterized by refinements of surjective submersions. That is, if  $\zeta : Y' \rightarrow Y$  is a refinement of a surjective submersion  $\pi : Y \rightarrow M$  by  $\pi' : Y' \rightarrow M$ , then  $F(\zeta) : S(\pi) \rightarrow S(\pi')$  is a 2-functor. Further, we require that  $F(\zeta' \circ \zeta) = F(\zeta') \circ F(\zeta)$  for iterated refinements. In this situation, one can form the *direct limit 2-category*

$$S_M := \varinjlim_{\pi} S(\pi).$$

We shall briefly describe a concrete model for this 2-category, the so-called Grothendieck construction. The precise form can be deduced from the general colimit description in  $\infty$ -categories; see [Lur09, Corollary 3.3.4.6]. An object of  $S_M$  is a pair  $(\pi, X)$  consisting of a surjective submersion  $\pi : Y \rightarrow M$  and an object  $X$  in  $S(\pi)$ . A *common refinement* of surjective submersions  $\pi_1, \pi_2$  is a commutative diagram

$$\begin{array}{ccc} & Z & \\ y_1 \swarrow & \downarrow \zeta & \searrow y_2 \\ Y_1 & & Y_2 \\ \pi_1 \searrow & \downarrow & \swarrow \pi_2 \\ & M & \end{array}$$

in which all maps are surjective submersions. A 1-morphism between objects  $(\pi_1, X_1)$  and  $(\pi_2, X_2)$  is a common refinement  $\zeta$  together with a 1-morphism

$$f : F(y_1)(X_1) \rightarrow F(y_2)(X_2)$$

in  $S(\zeta)$ . The composition of two 1-morphisms

$$(\zeta_{12}, f_{12}) : (\pi_1, X_1) \rightarrow (\pi_2, X_2) \quad \text{and} \quad (\zeta_{23}, f_{23}) : (\pi_2, X_2) \rightarrow (\pi_3, X_3)$$

is defined as follows. We consider the fibre product  $Z_{13} := Z_{12} \times_{Y_2} Z_{23}$  as a common refinement  $\zeta_{13} : Z_{13} \rightarrow M$  of  $\pi_1$  and  $\pi_3$ . Then, we set

$$(\zeta_{23}, f_{23}) \circ (\zeta_{12}, f_{12}) := (\zeta_{13}, F(\text{pr}_{Z_{12}})(f_{23}) \circ F(\text{pr}_{Z_{23}})(\zeta_{12})).$$

In order to define 2-morphisms between 1-morphisms  $(\zeta, f)$  and  $(\zeta', f')$  we consider pairs  $(\omega, \alpha)$  of a common refinement  $\omega : W \longrightarrow M$  of  $\zeta$  and  $\zeta'$  together with a 2-morphism

$$\alpha : F(z)(f) \Longrightarrow F(z')(f')$$

in  $S(\omega)$ , where  $z : W \longrightarrow Z$  and  $z' : W \longrightarrow Z'$  are the two refinement maps. A 2-morphism is then an equivalence class of pairs  $(\omega, \alpha)$ , where two pairs  $(\omega_1, \alpha_1)$  and  $(\omega_2, \alpha_2)$  are identified if the 2-morphisms agree when pulled back to the fibre product  $W_1 \times_{Z \times_M Z'} W_2$ .

In the present situation, we form the direct limits

$$\mathrm{Triv}^2(i)_M := \varinjlim_{\pi} \mathrm{Triv}_{\pi}^2(i) \quad \text{and} \quad \mathfrak{Des}^2(i)_M := \varinjlim_{\pi} \mathfrak{Des}_{\pi}^2(i).$$

One checks by inspection that the 2-functor  $\mathrm{Ex}_{\pi}$  commutes with the restriction 2-functors  $\mathrm{rec}_{\xi}$  of (4.2.2), so that it induces a 2-functor

$$\mathrm{Ex} : \mathrm{Triv}^2(i)_M \longrightarrow \mathfrak{Des}^2(i)_M. \tag{4.2.3}$$

Since limits preserve equivalences, we conclude from Proposition 4.1.1:

**Proposition 4.2.1.** *The 2-functor (4.2.3) is an equivalence of 2-categories.*

Now we look at the full sub-2-category  $\mathrm{Funct}_i(\mathcal{P}_2(M), T)$  of  $\mathrm{Funct}(\mathcal{P}_2(M), T)$  over those 2-functors  $F : \mathcal{P}_2(M) \longrightarrow T$  that admit a  $\pi$ -local  $i$ -trivialization, for some surjective submersion  $\pi : Y \longrightarrow M$ . We have a 2-functor

$$v : \mathrm{Triv}^2(i)_M \longrightarrow \mathrm{Funct}_i(\mathcal{P}_2(M), T)$$

which simply forgets the local trivialization which is attached to the objects on the left hand side. The 2-functor  $v$  is obviously an equivalence of 2-categories, since it is essentially surjective and the identity on Hom-categories. Summarizing, we get:

**Theorem 4.2.2.** *There is an equivalence*

$$\mathfrak{Des}^2(i)_M \cong \mathrm{Funct}_i(\mathcal{P}_2(M), T)$$

*between the 2-category of descent data and the 2-category of locally  $i$ -trivializable 2-functors, realized by a span of equivalences of 2-categories.*

Theorem 4.2.2 is the main result of this article. In [SW] we restrict it to an equivalence between important sub-2-categories: the one of *smooth descent data* (on the left hand side), and the one of *transport 2-functors* (on the right hand side). Transport 2-functors are an axiomatic formulation of connections on non-abelian gerbes.

## A Basic 2-Category Theory

We introduce notions and facts that we need in this article. For a more complete introduction to 2-categories, see e.g. [Lei].

**Definition A.1.** A (small) 2-category consists of a set of objects, for each pair  $(X, Y)$  of objects a set of 1-morphisms denoted  $f : X \longrightarrow Y$  and for each pair  $(f, g)$  of 1-morphisms  $f, g : X \longrightarrow Y$  a set of 2-morphisms denoted  $\varphi : f \Longrightarrow g$ , together with the following structure:

1. For every pair  $(f, g)$  of 1-morphisms  $f : X \longrightarrow Y$  and  $g : Y \longrightarrow Z$ , a 1-morphism  $g \circ f : X \longrightarrow Z$ , called the composition of  $f$  and  $g$ .
2. For every triple  $(f, g, h)$  of 1-morphisms  $f : W \longrightarrow X$ ,  $g : X \longrightarrow Y$  and  $h : Y \longrightarrow Z$ , a 2-morphism

$$a_{f,g,h} : (h \circ g) \circ f \Longrightarrow h \circ (g \circ f)$$

called the associator of  $f$ ,  $g$  and  $h$ .

3. For every object  $X$ , a 1-morphism  $\text{id}_X : X \longrightarrow X$ , called the identity 1-morphism of  $X$ .
4. For every 1-morphism  $f : X \longrightarrow Y$ , 2-morphisms  $l_f : f \circ \text{id}_X \Longrightarrow f$  and  $r_f : \text{id}_Y \circ f \Longrightarrow f$ , called the left and the right unifier.
5. For every pair  $(\varphi, \psi)$  of 2-morphisms  $\varphi : f \Longrightarrow g$  and  $\psi : g \Longrightarrow h$ , a 2-morphism  $\psi \bullet \varphi : f \Longrightarrow h$ , called the vertical composition of  $\varphi$  and  $\psi$ .
6. For every 1-morphism  $f$ , a 2-morphism  $\text{id}_f : f \Longrightarrow f$ , called the identity 2-morphism of  $f$ .
7. For every triple  $(X, Y, Z)$  of objects, 1-morphisms  $f, f' : X \longrightarrow Y$  and  $g, g' : Y \longrightarrow Z$ , and every pair  $(\varphi, \psi)$  of 2-morphisms  $\varphi : f \Longrightarrow f'$  and  $\psi : g \Longrightarrow g'$ , a 2-morphism  $\psi \circ \varphi : g \circ f \Longrightarrow g' \circ f'$ , called the horizontal composition of  $\varphi$  and  $\psi$ .

This structure has to satisfy the following list of axioms:

(C1) The vertical composition of 2-morphisms is associative,

$$(\phi \bullet \varphi) \bullet \psi = \phi \bullet (\varphi \bullet \psi)$$

whenever these compositions are well-defined, while the horizontal composition is compatible with the associator in the sense that the diagram

$$\begin{array}{ccc} (h \circ g) \circ f & \xRightarrow{(\psi \circ \varphi) \circ \phi} & (h' \circ g') \circ f' \\ \Downarrow a_{f,g,h} & & \Downarrow a_{f',g',h'} \\ h \circ (g \circ f) & \xRightarrow{\psi \circ (\varphi \circ \phi)} & h' \circ (g' \circ f') \end{array}$$

is commutative.

(C2) The identity 2-morphisms are units with respect to vertical composition,

$$\varphi \bullet \text{id}_f = \text{id}_g \bullet \varphi$$

for every 2-morphism  $\varphi : f \Rightarrow g$ , while the identity 1-morphisms are compatible with the unifiers and the associator in the sense that the diagram

$$\begin{array}{ccc} (g \circ \text{id}_Y) \circ f & \xRightarrow{a_{f,\text{id}_Y,g}} & g \circ (\text{id}_Y \circ f) \\ \searrow l_g \circ \text{id}_f & & \swarrow \text{id}_g \circ r_f \\ & g \circ f & \end{array}$$

is commutative. Horizontal composition preserves the identity 2-morphisms in the sense that

$$\text{id}_g \circ \text{id}_f = \text{id}_{g \circ f}.$$

(C3) Horizontal and vertical compositions are compatible in the sense that

$$(\psi_1 \bullet \psi_2) \circ (\varphi_1 \bullet \varphi_2) = (\psi_1 \circ \varphi_1) \bullet (\psi_2 \circ \varphi_2)$$

whenever these compositions are well-defined.

(C4) All associators and unifiers are invertible 2-morphisms and natural in  $f$ ,  $g$  and  $h$ , and the associator satisfies the pentagon axiom

$$\begin{array}{ccc}
 & ((k \circ h) \circ g) \circ f & \\
 \swarrow a_{g,h,k} \circ \text{id}_f & & \searrow a_{f,g,k \circ h} \\
 (k \circ (h \circ g)) \circ f & & (k \circ h) \circ (g \circ f) \\
 \searrow a_{f,h \circ g,k} & & \swarrow a_{g \circ f,h,k} \\
 k \circ ((h \circ g) \circ f) & \xrightarrow{\text{id}_k \circ a_{f,g,h}} & k \circ (h \circ (g \circ f)).
 \end{array}$$

In (C4) we have called a 2-morphism  $\varphi : f \Rightarrow g$  *invertible* or *2-isomorphism*, if there exists a 2-morphism  $\psi : g \Rightarrow f$  such that  $\psi \bullet \varphi = \text{id}_f$  and  $\varphi \bullet \psi = \text{id}_g$ . The axioms imply a *coherence theorem*: all diagrams of 2-morphisms whose arrows are labelled by associators, right or left unifiers, and identity 2-morphisms, are commutative. A 2-category is called *strict*, if

$$(h \circ g) \circ f = h \circ (g \circ f) \quad \text{and} \quad a_{f,g,h} = \text{id}_{h \circ g \circ f}$$

for all triples  $(f, g, h)$  of composable 1-morphisms, and if

$$f \circ \text{id}_X = f = \text{id}_Y \circ f \quad \text{and} \quad r_f = l_f = \text{id}_f$$

for all 1-morphisms  $f$ . Strict 2-categories allow us to draw pasting diagrams, since multiple compositions of 1-morphisms are well-defined without putting brackets. Pasting diagrams are often more instructive than commutative diagrams of 2-morphisms. For an explicit discussion of the strict case the reader is referred to Appendix A.1 in [SW11].

**Example A.2.** Let  $\mathfrak{C}$  be a monoidal category, i.e. a category equipped with a functor  $\otimes : \mathfrak{C} \times \mathfrak{C} \rightarrow \mathfrak{C}$ , a distinguished object  $\mathbf{I}$  in  $\mathfrak{C}$ , a natural transformation  $\alpha$  with components

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z),$$

and natural transformations  $\rho$  and  $\lambda$  with components

$$\rho_X : \mathbf{I} \otimes X \rightarrow X \quad \text{and} \quad \lambda_X : X \otimes \mathbf{I} \rightarrow X$$

which are subject to the usual coherence conditions, see, e.g. [ML97]. The monoidal category  $\mathfrak{C}$  defines a 2-category  $\mathcal{BC}$  in the following way: it has a single object, the 1-morphisms are the objects of  $\mathfrak{C}$  and the 2-morphisms between two 1-morphisms  $X$  and  $Y$  are the morphisms  $f : X \longrightarrow Y$  in  $\mathfrak{C}$ . The composition of 1-morphisms and the horizontal composition is the tensor product  $\otimes$ , and the associator  $a_{X,Y,Z}$  is given by the component  $\alpha_{Z,Y,X}$ . The identity 1-morphism is the tensor unit  $\mathbf{I}$ , and the unifiers are given by the natural transformations  $\rho$  and  $\lambda$ . The vertical composition and the identity are just the ones of  $\mathfrak{C}$ . It is straightforward to check that axioms (C1) to (C4) are either satisfied due to the axioms of the category  $\mathfrak{C}$ , the functor  $\otimes$ , or the natural transformations  $\alpha$ ,  $\rho$  and  $\lambda$ , or due to the coherence axioms. The 2-category  $\mathcal{BC}$  is strict if and only if the monoidal category  $\mathfrak{C}$  is strict.

In any 2-category, a 1-morphism  $f : X \longrightarrow Y$  is called *invertible*, if there exists another 1-morphism  $g : Y \longrightarrow X$  together with natural 2-isomorphisms  $i : g \circ f \Longrightarrow \text{id}_X$  and  $j : \text{id}_Y \Longrightarrow f \circ g$  such that the diagrams

$$\begin{array}{ccc}
(f \circ g) \circ f \xrightarrow{j^{-1} \circ \text{id}_f} \text{id}_Y \circ f & & (g \circ f) \circ g \xrightarrow{i \circ \text{id}_g} \text{id}_X \circ g \\
\Downarrow a_{f,g,f} & & \Downarrow a_{g,f,g} \\
f \circ (g \circ f) & \text{and} & g \circ (f \circ g) \\
\Downarrow \text{id}_f \circ i & & \Downarrow \text{id}_g \circ j^{-1} \\
f \circ \text{id}_X \xrightarrow{l_f} f & & g \circ \text{id}_Y \xrightarrow{l_g} g \\
& & \Downarrow r_g
\end{array} \tag{A.1}$$

are commutative. Let us remark that neither in the strict nor in the general case the inverse 1-morphism  $g$  is uniquely determined. We call a triple  $(g, i, j)$  a *weak inverse* of  $f$ . By *1-isomorphism* we mean an invertible 1-morphism together with a weak inverse.

**Remark A.3.** Often a 2-category is called bicategory, while a strict 2-category is called 2-category. Invertible 1-morphisms are often called adjoint equivalences.

**Definition A.4.** A (strict) 2-category in which every 1-morphism and every 2-morphism is invertible, is called (strict) 2-groupoid.



The following definition generalizes the one of a functor between categories.

**Definition A.5.** Let  $S$  and  $T$  be two 2-categories. A 2-functor  $F : S \longrightarrow T$  assigns

1. an object  $F(X)$  in  $T$  to each object  $X$  in  $S$ ,
2. a 1-morphism  $F(f) : F(X) \longrightarrow F(Y)$  in  $T$  to each 1-morphism  $f : X \longrightarrow Y$  in  $S$ ,  
and
3. a 2-morphism  $F(\varphi) : F(f) \Longrightarrow F(g)$  in  $T$  to each 2-morphism  $\varphi : f \Longrightarrow g$  in  $S$ .

Furthermore, it has

- (a) a 2-isomorphism  $u_X : F(\text{id}_X) \Longrightarrow \text{id}_{F(X)}$  in  $T$  for each object  $X$  in  $S$ , and
- (b) a 2-isomorphism  $c_{f,g} : F(g) \circ F(f) \Longrightarrow F(g \circ f)$  in  $T$  for each pair of composable 1-morphisms  $f$  and  $g$  in  $S$ .

Four axioms have to be satisfied:

(F1) The vertical composition is respected in the sense that

$$F(\psi \bullet \varphi) = F(\psi) \bullet F(\varphi) \quad \text{and} \quad F(\text{id}_f) = \text{id}_{F(f)}$$

for all composable 2-morphisms  $\varphi$  and  $\psi$ , and any 1-morphism  $f$ .

(F2) The horizontal composition is respected in the sense that the diagram

$$\begin{array}{ccc} F(g) \circ F(f) & \xrightarrow{F(\psi) \circ F(\varphi)} & F(g') \circ F(f') \\ \Downarrow c_{f,g} & & \Downarrow c_{f',g'} \\ F(g \circ f) & \xrightarrow{F(\psi \circ \varphi)} & F(g' \circ f') \end{array}$$

is commutative for all horizontally composable 2-morphisms  $\varphi$  and  $\psi$ .

(F3) The compositor  $c_{f,g}$  is compatible with the associators of  $S$  and  $T$  in the sense that

the diagram

$$\begin{array}{ccc}
(F(h) \circ F(g)) \circ F(f) & \xrightarrow{a_{F(f), F(g), F(h)}} & F(h) \circ (F(g) \circ F(f)) \\
\downarrow c_{g, h} \circ \text{id}_{F(f)} & & \downarrow \text{id}_{F(h)} \circ c_{f, g} \\
F(h \circ g) \circ F(f) & & F(h) \circ F(g \circ f) \\
\downarrow c_{f, h \circ g} & & \downarrow c_{g \circ f, h} \\
F((h \circ g) \circ f) & \xrightarrow{F(a_{f, g, h})} & F(h \circ (g \circ f))
\end{array}$$

is commutative for all composable 1-morphisms  $f$ ,  $g$  and  $h$ .

(F4) Compositor and unitor are compatible with the unifiers of  $S$  and  $T$  in the sense that the diagrams

$$\begin{array}{ccc}
F(f) \circ F(\text{id}_X) & \xrightarrow{c_{\text{id}_X, f}} & F(f \circ \text{id}_X) \\
\downarrow \text{id}_{F(f)} \circ u_X & & \downarrow F(l_f) \\
F(f) \circ \text{id}_{F(X)} & \xrightarrow{l_{F(f)}} & F(f)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
F(\text{id}_Y) \circ F(f) & \xrightarrow{c_{f, \text{id}_Y}} & F(\text{id}_Y \circ f) \\
\downarrow u_Y \circ \text{id}_{F(f)} & & \downarrow F(r_f) \\
\text{id}_{F(Y)} \circ F(f) & \xrightarrow{r_{F(f)}} & F(f)
\end{array}$$

are commutative for every 1-morphism  $f$ .

Sometimes we represent a 2-functor  $F : S \rightarrow T$  diagrammatically as an assignment

$$F : \quad X \begin{array}{c} \xrightarrow{f} \\ \Downarrow \phi \\ \xrightarrow{g} \end{array} Y \quad \mapsto \quad F(X) \begin{array}{c} \xrightarrow{F(f)} \\ \Downarrow F(\phi) \\ \xrightarrow{F(g)} \end{array} F(Y) .$$

In case that the 2-category  $T$  is strict, and the axioms (F2) to (F4) can be expressed by pasting diagrams in the following way:

- Axioms (F2) is equivalent to the equality

$$\begin{array}{c}
\begin{array}{ccccc}
& F(f) & & F(Y) & \\
& \nearrow F(\phi) & & \nwarrow F(\psi) & \\
F(X) & & F(f') & & F(g') & \\
& \nwarrow F(\phi) & & \nearrow F(\psi) & \\
& F(g' \circ f') & & & 
\end{array} \\
= & 
\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
& \downarrow c_{f, g} & \\
F(X) & \xrightarrow{F(g \circ f)} & F(Z) \\
& \downarrow F(\psi \circ \phi) & \\
& F(g' \circ f') & 
\end{array}
\end{array}$$

- Axiom (F3) is equivalent to the tetrahedron identity

$$\begin{array}{ccc}
F(X) & \xrightarrow{F(g)} & F(Y) \\
\uparrow F(f) & \searrow c_{f,g} & \nearrow F(h) \\
& F(g \circ f) & \\
F(W) & \xrightarrow{F(h \circ g \circ f)} & F(Z)
\end{array}
=
\begin{array}{ccc}
F(X) & \xrightarrow{F(g)} & F(Y) \\
\uparrow F(f) & \searrow c_{h,g} & \nearrow F(h) \\
& F(h \circ g) & \\
F(W) & \xrightarrow{F(h \circ g \circ f)} & F(Z)
\end{array}$$

- Axiom (F4) is equivalent to the equalities

$$c_{\text{id}_X, f} = \text{id}_{F(f)} \circ u_X \quad \text{and} \quad c_{f, \text{id}_Y} = u_Y \circ \text{id}_{F(f)}.$$

We use two layers of strictness conditions for 2-functors, normally in a situation when  $S$  and  $T$  are strict 2-categories. Firstly, we call a 2-functor  $F : S \longrightarrow T$  *normalized* if

$$F(\text{id}_X) = \text{id}_{F(X)} \quad \text{and} \quad u_X = \text{id}_{\text{id}_{F(X)}} \quad (\text{A.2})$$

for all objects  $X$  in  $S$ , and if

$$F(g) \circ F(f) = \text{id}_{F(X)} \quad \text{and} \quad c_{f,g} = \text{id}_{\text{id}_X} \quad (\text{A.3})$$

for all 1-morphisms  $f : X \longrightarrow Y$  and  $g : Y \longrightarrow X$  such that  $g \circ f = \text{id}_X$ . Roughly speaking, normalized 2-functors strictly respect identities and inverses. The second and stronger requirement is that the 2-functor  $F : S \longrightarrow T$  is *strict*: this requires (A.2) while (A.3) is superseded by the condition that

$$F(g) \circ F(f) = F(g \circ f) \quad \text{and} \quad c_{f,g} = \text{id}_{F(g \circ f)}$$

for *all* composable 1-morphisms  $f$  and  $g$ . In case of strict 2-functors between strict 2-categories only axioms (F1) and (F2) remain, claiming that both compositions are respected.

The following definition generalizes a natural transformation between functors.

**Definition A.6.** Let  $F_1$  and  $F_2$  be two 2-functors from  $S$  to  $T$ . A pseudonatural transformation  $\rho : F_1 \longrightarrow F_2$  assigns

1. a 1-morphism  $\rho(X) : F_1(X) \longrightarrow F_2(X)$  in  $T$  to each object  $X$  in  $S$ , and

2. a 2-isomorphism  $\rho(f) : \rho(Y) \circ F_1(f) \Longrightarrow F_2(f) \circ \rho(X)$  in  $T$  to each 1-morphism  $f : X \longrightarrow Y$  in  $S$ ,

such that two axioms are satisfied:

(T1) The composition of 1-morphisms in  $S$  is respected in the sense that the diagram

$$\begin{array}{ccc}
(\rho(Z) \circ F_1(g)) \circ F_1(f) & \xRightarrow{a_{F_1(f), F_1(g), \rho(Z)}} & \rho(Z) \circ (F_1(g) \circ F_1(f)) \\
\downarrow \rho(g) \circ \text{id}_{F_1(f)} & & \downarrow \text{id}_{\rho(Z)} \circ (c_1)_{f,g} \\
(F_2(g) \circ \rho(Y)) \circ F_1(f) & & \rho(Z) \circ F_1(g \circ f) \\
\downarrow a_{F_1(f), \rho(Y), F_2(g)} & & \downarrow \rho(g \circ f) \\
F_2(g) \circ (\rho(Y) \circ F_1(f)) & & F_2(g \circ f) \circ \rho(X) \\
\downarrow \text{id}_{F_2(g)} \circ \rho(f) & & \downarrow (c_2)_{f,g}^{-1} \circ \text{id}_{\rho(X)} \\
F_2(g) \circ (F_2(f) \circ \rho(X)) & \xRightarrow{a_{\rho(X), F_2(f), F_2(g)}^{-1}} & (F_2(g) \circ F_2(f)) \circ \rho(X)
\end{array}$$

is commutative for all composable 1-morphisms  $f$  and  $g$ . Here,  $a$  is the associator of the 2-category  $T$  and  $c_1$  and  $c_2$  are the compositors of the 2-functors  $F_1$  and  $F_2$ , respectively.

(T2) It is natural in the sense that the diagram

$$\begin{array}{ccc}
\rho(Y) \circ F_1(f) & \xRightarrow{\rho(f)} & F_2(f) \circ \rho(X) \\
\downarrow \text{id}_{\rho(Y)} \circ F_1(\varphi) & & \downarrow F_2(\varphi) \circ \text{id}_{\rho(X)} \\
\rho(Y) \circ F_1(g) & \xRightarrow{\rho(g)} & F_2(g) \circ \rho(X)
\end{array}$$

is commutative for all 2-morphisms  $\varphi : f \Longrightarrow g$ .

If one considers a version of pseudonatural transformations where the 2-morphisms  $\rho(f)$  do not have to be invertible, there is a third axiom related to the value of  $\rho$  at the identity 1-morphism  $\text{id}_X$  of an object  $X$  in  $S$ . In our setup this axiom is automatically satisfied, as the following lemma shows.

**Lemma A.7.** *Let  $\rho : F_1 \longrightarrow F_2$  be a pseudonatural transformation between 2-functors  $F_1$*

and  $F_2$  with unitors  $u^1$  and  $u^2$ , respectively. Then, the following assertions hold.

(i) The diagram

$$\begin{array}{ccc}
\rho(X) \circ F_1(\text{id}_X) & \xrightarrow{\rho(\text{id}_X)} & F_2(\text{id}_X) \circ \rho(X) \\
\text{id}_{\rho(X)} \circ u_X^1 \Downarrow & & \Downarrow u_X^2 \circ \text{id}_{\rho(X)} \\
\rho(X) \circ \text{id}_{F_1(X)} & \xrightarrow{l_{\rho(X)}} \rho(X) \xrightarrow{r_{\rho(X)}^{-1}} & \text{id}_{F_2(X)} \circ \rho(X)
\end{array}$$

is commutative.

(ii) If  $S$  and  $T$  are strict 2-categories, and  $F_1$  and  $F_2$  are normalized 2-functors, then

$$\rho(\text{id}_X) = \text{id}_{\rho(X) \circ F_1(\text{id}_X)} = \text{id}_{F_2(\text{id}_X) \circ \rho(X)}$$

for all objects  $X$  in  $S$ , and

$$\rho(g) \circ \text{id}_{F_1(f)} = \text{id}_{F_2(g)} \circ \rho(f)^{-1}$$

for all 1-morphisms  $f, g$  in  $S$  with  $g \circ f = \text{id}$ .

Proof. For (i) one applies axiom (T1) to 1-morphisms  $f = g = \text{id}_X$ . Then one uses axiom (T2) for  $\rho$ , axiom (F4) for both 2-functors, axiom (C2) for  $T$ , and the invertibility of the 2-morphism  $\rho(g)$  and of the 1-morphism  $F_2(\text{id}_X)$ . The first claim of (ii) follows from (i) under the strictness assumptions, and the second claim follows from the first claim and axiom (T1).  $\square$

Sometimes we represent a pseudonatural transformation  $\rho : F_1 \rightarrow F_2$  diagrammatically by

$$\rho : X \xrightarrow{f} Y \quad \mapsto \quad \begin{array}{ccc} F_1(X) & \xrightarrow{F_1(f)} & F_1(Y) \\ \rho(X) \downarrow & \swarrow \rho(f) & \downarrow \rho(Y) \\ F_2(X) & \xrightarrow{F_2(f)} & F_2(Y) \end{array}$$

and if the 2-category  $T$  is strict, the axioms can be expressed by pasting diagrams in the following way:

- Axiom (T1) is equivalent to

$$\begin{array}{ccc}
 F_1(X) & \xrightarrow{F_1(f)} & F_1(Y) & \xrightarrow{F_1(g)} & F_1(Z) \\
 \rho(X) \downarrow & \swarrow \rho(f) & \downarrow \rho(Y) & \swarrow \rho(g) & \downarrow \rho(Z) \\
 F_2(X) & \xrightarrow{F_2(f)} & F_2(Y) & \xrightarrow{F_2(g)} & F_2(Z) \\
 & \searrow & \downarrow (c_2)_{f,g} & \searrow & \\
 & & F_2(g \circ f) & & 
 \end{array}
 =
 \begin{array}{ccc}
 & F_1(g) \circ F_1(f) & \\
 & \downarrow (c_1)_{f,g} & \\
 F_1(X) & \xrightarrow{F_1(g \circ f)} & F_1(Z) \\
 \rho(X) \downarrow & \swarrow \rho(g \circ f) & \downarrow \rho(Z) \\
 F_2(X) & \xrightarrow{F_2(g \circ f)} & F_2(Z)
 \end{array}$$

- Axiom (T2) is equivalent to

$$\begin{array}{ccc}
 F_1(X) & \xrightarrow{F_1(f)} & F_1(Y) \\
 \rho(X) \downarrow & \swarrow \rho(f) & \downarrow \rho(Y) \\
 F_2(X) & \xrightarrow{F_2(f)} & F_2(Y) \\
 & \searrow & \downarrow F_2(\varphi) \\
 & & F_2(g)
 \end{array}
 =
 \begin{array}{ccc}
 & F_1(f) & \\
 & \downarrow F_1(\varphi) & \\
 F_1(x) & \xrightarrow{F_1(g)} & F_1(Y) \\
 \rho(X) \downarrow & \swarrow \rho(g) & \downarrow \rho(Y) \\
 F_2(X) & \xrightarrow{F_2(g)} & F_2(Y)
 \end{array}$$

We need one more definition for situations where two pseudonatural transformations are present.

**Definition A.8.** Let  $F_1, F_2 : S \longrightarrow T$  be two 2-functors and let  $\rho_1, \rho_2 : F_1 \longrightarrow F_2$  be pseudonatural transformations. A modification  $\mathcal{A} : \rho_1 \Longrightarrow \rho_2$  assigns a 2-morphism

$$\mathcal{A}(X) : \rho_1(X) \Longrightarrow \rho_2(X)$$

in  $T$  to any object  $X$  in  $S$ , such that the diagram

$$\begin{array}{ccc}
 \rho_1(Y) \circ F_1(f) & \xRightarrow{\rho_1(f)} & F_2(f) \circ \rho_1(X) \\
 \mathcal{A}(Y) \circ \text{id}_{F_1(f)} \Downarrow & & \Downarrow \text{id}_{F_2(f)} \circ \mathcal{A}(X) \\
 \rho_2(Y) \circ F_1(f) & \xRightarrow{\rho_2(f)} & F_2(f) \circ \rho_2(X)
 \end{array} \tag{A.4}$$

is commutative for every 1-morphism  $f$  in  $S$ .

In the case that  $T$  is a strict 2-category, the latter diagram is equivalent to a pasting diagram, see Definition A.4 in [SW11].

As one might expect, 2-functors, pseudonatural transformations, and modifications fit into the structure of a 2-category that we denote by  $\text{Func}(S, T)$ . It is strict if and only if  $T$  is strict. Note that the definition of invertibility in a 2-category applies; we call a 2-isomorphism in the 2-category  $\text{Func}(S, T)$  *invertible modification*, and a 1-isomorphism *pseudonatural equivalence*. This leads to the following

**Definition A.9.** *Let  $S$  and  $T$  be 2-categories. A 2-functor  $F : S \longrightarrow T$  is called an equivalence of 2-categories, if there exists a 2-functor  $G : T \longrightarrow S$  together with pseudonatural equivalences  $\rho_S : G \circ F \longrightarrow \text{id}_S$  and  $\rho_T : F \circ G \longrightarrow \text{id}_T$ .*

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